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Quelques liens entre la théorie de l'intégration non-additive et les domaines de la finance et de l'assurance

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dirigée par Marie-Claire QUENEZ

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Résumé

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Cette thèse de doctorat fait quelques liens entre la théorie de l'intégration non-additive et les notions d'ordre stochastique et de mesure du risque utilisées en finance et en assurance. Nous faisons un usage extensif des fonctions d'ensembles monotones normalisées, appelées également capacités, ou encore probabilités non-additives, ainsi que des intégrales qui leur sont associées, appelées intégrales de Choquet.
Dans le premier chapitre, nous établissons une généralisation des inégalités de Hardy-Littlewood au cas d'une capacité. Nous y faisons également quelques compléments sur les fonctions quantiles par rapport à une capacité.
Dans le deuxième chapitre, nous généralisons la notion de dominance stochastique croissante convexe au cas d'une capacité, et nous résolvons un problème d'optimisation dont la contrainte est donnée en termes de cette relation généralisée.
Dans le troisième chapitre, nous nous intéressons à la généralisation de la notion de dominance stochastique croissante. Nous étudions également les classes de mesures du risque satisfaisant aux propriétés d'additivité comonotone et de consistance par rapport à l'une des relations de dominance stochastique "généralisées" précédemment considérées. Nous caractérisons ces mesures du risque en termes d'intégrales de Choquet par rapport à une "distorsion" de la capacité initiale.
Le quatrième chapitre est consacré à des mesures du risque admettant une représentation "robuste" en termes de maxima (sur un ensemble de fonctions de distorsion) d'intégrales de Choquet par rapport à des capacités distordues.

Some links between the non-additive integration theory and the fields of finance and insurance

Abstract

In this dissertation we establish some links between the non-additive integration theory and some useful notions in finance and insurance, such as the notions of stochastic ordering and risk measure.

In the framework of ambiguity, the notion of capacity (or non-additive probability) replaces that of probability measure, and Choquet integrals replace the usual mathematical expectations. In this thesis, we extend the notions of increasing, and increasing convex stochastic dominance, well-known in the case of a probability, to this more general framework. We characterize these relations in terms of distribution functions and quantile functions with respect to the initial capacity. We also establish a generalization of Hardy-Littlewood's inequalities to the case of a capacity, which we apply in solving an optimization problem whose constrains are given by means of the "generalized" increasing convex relation.

We are then interested in the classes of monetary risk measures having the properties of comonotonic additivity and consistency with respect to a given "generalized" stochastic dominance relation. These are characterized in terms of Choquet integrals with respect to a distorted capacity. A Kusuoka-type characterization of the class of risk measures having the properties of comonotonic additivity and consistency with respect to the "generalized" increasing convex ordering is also established. Finally, we are interested in those risk measures that have a "robust" representation as a maximum, over a set of distortion functions, of Choquet integrals with respect to a distortion of the initial capacity.

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Notations

$\mathbb{R}$	the real numbers
$\bar{\mathbb{R}}$	the extended real numbers
$\mathbb{R}_+$	the non-negative real numbers
x_+	positive part of the real number x, i.e. $x_+ := \max(x, 0)$
x_-	negative part of the real number x, i.e. $x_- := \max(-x, 0)$
$(\Omega, \mathcal{F})$	a measurable space
$\mathcal{P}(\Omega)$	the power set of Ω, i.e. the collection of all subsets of Ω
A^c	the complement of a set A in Ω (i.e. $A^c := \{\omega \in \Omega : \omega \notin A\}$)
χ	the space of bounded real-valued measurable functions on $(\Omega, \mathcal{F})$
χ_+	the set of non-negative elements of χ
$P, Q, ...$	probability measures
$L^\infty(\Omega, \mathcal{F}, P)$	the space of P-essentially bounded random variables on $(\Omega, \mathcal{F})$
$\mu, \nu, ...$	capacities
$\mathbb{E}_\mu(\cdot)$	Choquet integral with respect to the capacity μ
X, Y, Z	measurable functions on $(\Omega, \mathcal{F})$
$r^+_{X,\mu}$	upper quantile function of X with respect to the capacity μ
$r^-_{X,\mu}$	lower quantile function of X with respect to the capacity μ
$\square$	this symbol denotes the end of a proof

Introduction générale

Le travail présenté dans cette thèse de doctorat se situe dans l'interface entre les domaines de la finance et de l'assurance, d'une part, et celui des probabilités, d'autre part. Plus concrètement, dans ce travail nous faisons quelques liens entre la théorie de l'intégration non-additive et des notions utilisées en finance et en assurance, telles que les notions d'ordre stochastique et de mesure du risque (cette dernière étant également connue sous le nom de principe de calcul de primes en assurance).
Notons que la quantification des risques est un sujet d'actualité dont l'importance a été mise en évidence, en particulier, par la crise financière actuelle.
D'autre part, rappelons que les questions liées à la prise en compte et à la modélisation des situations d'ambiguïté ont attiré un grand intérêt, que ce soit dans le domaine de l'économie mathématique, ou, plus récemment, dans le domaine de la finance mathématique (avec, par exemple, les problèmes d'incertitude de modèle). Dans le domaine de la théorie de la décision, le cadre axiomatique "classique" de Von Neumann et Morgenstern, permettant de représenter les préférences d'un agent économique en termes de l'espérance de son utilité par rapport à une probabilité, a été remis en question par des études empiriques et des paradoxes (tels que le paradoxe d'Allais ou le paradoxe d'Ellsberg). Des théories alternatives à la théorie "classique" ont été proposées, dans lesquelles apparaissent des fonctions d'ensembles plus générales qu'une probabilité, et des "espérances" par rapport à ces fonctions d'ensembles (en d'autres termes, des capacités et des intégrales de Choquet). Nous reviendrons sur certaines de ces théories et sur les outils mathématiques qu'elles utilisent dans la suite de cette introduction.

0.1 Capacités et intégrales de Choquet

Dans la suite le terme capacité sera utilisé dans le sens suivant :

Définition 0.1.1 *Soit* $(\Omega, \mathcal{F})$ *un espace mesurable. Une fonction d'ensembles* $\mu : \mathcal{F} \longrightarrow [0,1]$ *est appelée* capacité *si elle satisfait aux propriétés suivantes :*

(i) $\mu(\varnothing) = 0$

(ii) *(monotonie)* $A, B \in \mathcal{F}, A \subset B \Rightarrow \mu(A) \leq \mu(B)$.

(iii) *(normalisation)* $\mu(\Omega) = 1$

Nous pouvons voir la notion de capacité comme une généralisation de celle de mesure de probabilité. Une capacité n'est pas nécessairement additive. L'intérêt particulier porté aux fonctions d'ensembles non-additives remonte au traité intitulé "Theory of capacities" du mathématicien français Gustave Choquet (Choquet 1954). Nous attirons l'attention du lecteur sur la remarque suivante :
Dans le travail de Choquet le cadre retenu est un cadre topologique et le terme *capacités* (positives) se rapporte à des fonctions d'ensembles qui ne sont pas nécessairement normalisées, et qui, en plus de la propriété de monotonie (ii) susmentionnée, vérifient des propriétés supplémentaires.

En suivant Choquet, l'intégrale par rapport à une capacité (où le terme capacité est entendu au sens de la définition ci-dessus) d'une fonction mesurable positive a été définie de la manière suivante :

Définition 0.1.2 *Soit X une fonction mesurable définie sur $(\Omega, \mathcal{F})$ à valeurs dans $\bar{\mathbb{R}}_+$. Soit μ une capacité sur $(\Omega, \mathcal{F})$. L'intégrale de Choquet $\mathbb{E}_\mu(X)$ de X par rapport à la capacité μ est définie par*

$$\mathbb{E}_\mu(X) := \int_0^{+\infty} \mu(X > x)dx.$$

La définition 0.1.2 ci-dessus généralise une formule bien connue en théorie de la mesure : dans le cas particulier où la capacité μ est une mesure de probabilité, l'intégrale de Choquet coïncide avec la notion usuelle d'intégrale de Lebesgue d'une variable positive par rapport à une probabilité.
La notion d'intégrale de Choquet a été ensuite étendue à des fonctions mesurables pouvant prendre des valeurs négatives (nous renvoyons le lecteur aux monographies de Denneberg 1994 et de Pap 1995 pour une présentation de ces extensions et pour des références). Le présentation adoptée dans cette introduction est inspirée de Denneberg (1994).

Nous rappelons d'abord la définition suivante :

Définition 0.1.3 *Soit μ une capacité sur $(\Omega, \mathcal{F})$. La capacité duale $\bar{\mu}$ de la capacité μ est définie par $\bar{\mu}(A) := 1 - \mu(A^c)$, pour tout $A \in \mathcal{F}$.*

Dans le cas particulier où μ est une probabilité, la duale $\bar{\mu}$ est également une probabilité, et nous avons l'égalité $\bar{\mu} = \mu$.
Pour une fonction mesurable X sur $(\Omega, \mathcal{F})$ à valeurs dans $\bar{\mathbb{R}}$, nous notons par X_+ la partie positive de X ($X_+ := \max(X, 0)$) et par X_- la partie négative de X ($X_- := \max(-X, 0)$).
Deux extensions de la définition 0.1.2 à des fonctions mesurables pouvant prendre des

valeurs négatives ont été proposées dans la littérature. Nous rappelons d'abord la définition de l'intégrale *symétrique* (ou intégrale de *Šipoš*) de X par rapport à une capacité μ :

Définition 0.1.4 *Soit X une fonction mesurable définie sur $(\Omega, \mathcal{F})$ à valeurs dans $\mathbb{R}$. Soit μ une capacité sur $(\Omega, \mathcal{F})$. L'intégrale* symétrique *(ou intégrale de Šipoš) $\mathbb{E}_\mu^S(X)$ de X par rapport à la capacité μ est définie par*

$$\mathbb{E}_\mu^S(X) := \mathbb{E}_\mu(X_+) - \mathbb{E}_\mu(X_-),$$

pourvu que le terme de droite ait un sens.

L'intégrale de *Šipoš* ci-dessus coïncide avec l'intégrale de la définition 0.1.2 pour des fonctions mesurables X à valeurs dans $\mathbb{R}_+$. Dans le cas particulier où la capacité μ est une probabilité, la définition ci-dessus coïncide avec la définition usuelle de l'intégrale de Lebesgue d'une variable aléatoire X par rapport à une probabilité (cf. sous-section II.3 dans Neveu 1970).

Une autre possibilté pour étendre l'intégrale de la définition 0.1.2 à des fonctions mesurables pouvant prendre des valeurs négatives est la suivante :

Définition 0.1.5 *Soit X une fonction mesurable définie sur $(\Omega, \mathcal{F})$ à valeurs dans $\mathbb{R}$. Soit μ une capacité sur $(\Omega, \mathcal{F})$. L'intégrale* asymétrique *(ou* intégrale de Choquet*) $\mathbb{E}_\mu(X)$ de X par rapport à la capacité μ est définie par*

$$\mathbb{E}_\mu(X) := \mathbb{E}_\mu(X_+) - \mathbb{E}_{\bar{\mu}}(X_-),$$

pourvu que le terme de droite ait un sens.

Dans le cas particulier où la capacité μ est une probabilité, la définition ci-dessus coïncide avec la définition usuelle de l'intégrale de Lebesgue d'une variable aléatoire X par rapport à une probabilité. Comme indiqué dans la définition 0.1.5 ci-dessus, c'est l'intégrale de cette définition qui est connue sous le nom d'intégrale de Choquet. Cette intégrale a trouvé de nombreuses applications, en particulier dans les domaines de la théorie des jeux, de l'économie, de la finance et de l'assurance. Nous en rappellerons quelques-unes dans la suite de cette introduction. Nous pouvons démontrer (cf. Denneberg 1994, pages 62 et 87) que

$$\mathbb{E}_\mu(X) = \int_0^{+\infty} \mu(X > x)dx + \int_{-\infty}^0 (\mu(X > x) - 1)dx, \qquad (0.1.1)$$

où les intégrales du terme de droite sont des intégrales de Riemann généralisées. L'expression (0.1.1) est souvent retenue comme définition de l'intégrale de Choquet (cf., par exemple, Föllmer et Schied (2004)).

Rappelons quelques définitions dont nous aurons besoin dans la suite :

– Une capacité μ est dite *convexe* (ou surmodulaire) si

$$A, B \in \mathcal{F} \Rightarrow \mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B).$$

– Une capacité μ est dite *concave* (ou sous-modulaire, ou alternante d'ordre 2, ou fortement sous-additive) si

$$A, B \in \mathcal{F} \Rightarrow \mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B).$$

– Une capacité μ est dite *"continue par en-dessous"* si elle satisfait à la propriété suivante de continuité séquentielle le long des suites croissantes d'événements :

$$(A_n) \subset \mathcal{F} \text{ telle que } A_n \ \subset A_{n+1}, \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\cup_{n=1}^{\infty} A_n).$$

Certains auteurs (cf. Dellacherie 1971) utilisent la terminologie : la capacité "monte".

– Une capacité μ est dite *"continue par en-dessus"* si elle satisfait à la propriété suivante de continuité séquentielle le long des suites décroissantes d'événements :

$$(A_n) \subset \mathcal{F} \text{ telle que } A_n \supset A_{n+1}, \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n).$$

Certains auteurs utilisent la terminologie : la capacité "descend".

Nous notons que la propriété de concavité (ou sous-modularité, ou sous-additivité forte) d'une capacité μ implique la propriété de sous-additivité de μ, ce qui explique la terminologie *sous-additivité forte* utilisée par Choquet (1954).

Nous rappelons la notion de probabilité distordue qui a trouvé de multiples applications en économie, finance et assurance. Parmi les nombreux travaux utilisant cette notion nous pouvons citer les travaux de Wang (1996), Wang et al. (1997), Denneberg (1990) dans le domaine de l'assurance, Yaari (1997), Quiggin (1982), Tversky et Kahneman (1992) dans le domaine de la théorie de la décision, plus récemment les travaux de Carlier et Dana (2003), Carlier et Dana (2006), Carlier et Dana (2011) dans le domaine de l'économie mathématique, les travaux de Jin et Zhou (2008) dans le domaine de la gestion opimale de portefeuille, etc.

Définition/Proposition 0.1.1 *Soit P une mesure de probabilité sur $(\Omega, \mathcal{F})$. Soit $\psi : [0,1] \to [0,1]$ une fonction croissante sur $[0,1]$ telle que $\psi(0) = 0$ et $\psi(1) = 1$. La fonction d'ensembles $\psi \circ P$ définie par $\psi \circ P(A) := \psi(\mu(A)), \forall A \in \mathcal{F}$, est une capacité au sens de la définition 0.1.1. La fonction ψ est appelée fonction de distorsion et la capacité $\psi \circ P$ est appelée probabilité distordue. Si la fonction de distorsion ψ est concave, la capacité $\psi \circ P$ est une capacité concave (ou sous-modulaire).*

Ainsi, la notion de capacité concave (ou sous-modulaire) pourrait-elle être vue comme une généralisation de la notion de probabilité distordue avec fonction de distorsion concave.

La notion d'intégrale de Choquet est liée à la notion de fonctions mesurables comonotones que nous rappelons :

Définition 0.1.6 *Deux fonctions mesurables à valeurs réelles X et Y sur $(\Omega, \mathcal{F})$ sont dites* comonotones *si*

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0,\ \forall (\omega, \omega') \in \Omega \times \Omega.$$

Nous rappelons quelques propriétés basiques de l'intégrale de Choquet (cf. proposition 5.1 de Denneberg 1994) où nous faisons la convention que les propriétés sont valides pourvu que les expressions aient un sens.

Proposition 0.1.1 *Soit μ une capacité sur $(\Omega, \mathcal{F})$. Soit X et Y deux fonctions mesurables à valeurs réelles sur $(\Omega, \mathcal{F})$. Nous avons les propriétés suivantes :*

- *(homogénéité positive)* $\mathbb{E}_\mu(\lambda X) = \lambda \mathbb{E}_\mu(X), \forall \lambda \in \mathbb{R}_+$
- *(monotonie)* $X \leq Y \Rightarrow \mathbb{E}_\mu(X) \leq \mathbb{E}_\mu(Y)$
- *(invariance par translation)* $\mathbb{E}_\mu(X + b) = \mathbb{E}_\mu(X) + b, \forall b \in \mathbb{R}$
- *(asymétrie)* $\mathbb{E}_\mu(-X) = -\mathbb{E}_{\bar{\mu}}(X)$, *où $\bar{\mu}$ dénote la capacité duale de μ*
- *(additivité comonotone) Si X et Y sont comonotones, alors* $\mathbb{E}_\mu(X + Y) = \mathbb{E}_\mu(X) + \mathbb{E}_\mu(Y)$.

Si la capacité μ est concave (ou sous-modulaire), la fonctionnelle $\mathbb{E}_\mu(\cdot)$ est sous-additive.

Proposition 0.1.2 *Soit μ une capacité concave (ou sous-modulaire) sur $(\Omega, \mathcal{F})$. Soit X et Y deux fonctions mesurables à valeurs réelles sur $(\Omega, \mathcal{F})$ telles que $\mathbb{E}_\mu(X) > -\infty$ et $\mathbb{E}_\mu(Y) > -\infty$. Alors,*

$$\textit{(sous-additivité)} \qquad \mathbb{E}_\mu(X + Y) \leq \mathbb{E}_\mu(X) + \mathbb{E}_\mu(Y).$$

La propriété d'additivité comonotone de l'intégrale de Choquet a été mise en évidence par Dellacherie (1971). La propriété de sous-additivité de l'intégrale de Choquet $\mathbb{E}_\mu(\cdot)$ dans le cas où μ est une capacité concave a été mise en evidence par Choquet (1954). Ces propriétés ont trouvé des interprétations économiques dans la littérature relative au mesures du risque et nous y reviendrons dans la suite.

0.2 Espérance sous-linéaire et intégrale de Choquet

L'intégrale de Choquet par rapport à une capacité μ peut être vue comme une espérance non-additive. En vertu de la proposition 0.1.2 rappelée ci-dessus, cette espérance non-additive est sous-linéaire dans le cas où la capacité μ est concave (ou sous-modulaire). Ces observations pourraient amener le lecteur à s'interroger sur les liens entre l'intégrale de Choquet par rapport à une capacité μ concave, d'une part, et des espérances sous-linéaires de la forme $\sup_{P\in\mathcal{P}} \mathbb{E}_P(\cdot)$ où $\mathcal{P}$ est une famille non-vide de probabilités sur $(\Omega, \mathcal{F})$, d'autre part. Rappelons que des fonctionnelles de cette forme-ci apparaissent dans différents domaines de la finance et/ou des probabilités parmi lesquels nous pouvons citer la théorie d'évaluation d'actifs contingents en marché incomplet, la théorie des mesures du risque cohérentes, ou encore la G-analyse stochastique initiée récemment par S. Peng. Dans cette section de l'introduction nous faisons quelques rappels sur les liens entre ces deux notions d'espérance non-linéaire.

Plus précisement, nous nous donnons $\mathcal{P}$ une famille non-vide de probabilités sur $(\Omega, \mathcal{F})$. Nous remarquons que la fonction d'ensembles μ définie par

$$\mu(A) := \sup_{P\in\mathcal{P}} P(A), \ \forall A \in \mathcal{F},$$

est une capacité (au sens de la définition 0.1.1). La capacité μ est sous-additive sans être nécessairement concave (cf., par exemple, Huber et Strassen 1973). La capacité μ est continue par en-dessous mais elle n'est pas nécessairement continue par en-dessus.

Nous considérons les deux fonctionnelles $X \in \chi \mapsto \sup_{P\in\mathcal{P}} \mathbb{E}_P(X)$ et $X \in \chi \mapsto \mathbb{E}_{\sup_{P\in\mathcal{P}}}(X)$. Ces deux fonctionnelles coïncident sur l'ensemble des fonctions indicatrices $\{\mathbb{I}_A, A \in \mathcal{F}\}$. De plus, nous avons l'inégalité suivante (cf. proposition 5.2 de Denneberg 1994) :

$$\sup_{P\in\mathcal{P}} \mathbb{E}_P(X) \le \mathbb{E}_\mu(X), \text{ pour tout } X \in \chi. \tag{0.2.1}$$

Cependant, l'inégalité inverse n'a pas necéssairement lieu.

Rappelons également le résultat suivant (cf., par exemple, proposition 10.3 de Denneberg 1994, ou théorème 4.88 de Föllmer et Schied (2004), ainsi que les références données par ces auteurs). Le symbole $\mathcal{M}_{1,f}$ dénote l'ensemble des probabilités finiment additives sur $(\Omega, \mathcal{F})$, i.e. $\mathcal{M}_{1,f}$ est l'ensemble des fonctions d'ensembles finiment additives, positives et dont le poids total est égal à 1.

Théorème 0.2.1 *Soit ν une capacité sur $(\Omega, \mathcal{F})$.*
Les assertions suivantes sont équivalentes :

(i) *L'intégrale de Choquet $\mathbb{E}_\nu(\cdot)$ est sous-additive sur χ.*

(ii) *La capacité ν est concave.*

(iii) *L'ensemble core(ν) défini par core(ν) $:= \{Q \in \mathcal{M}_{1,f} : \bar{\mu}(A) \le Q(A) \le \nu(A), \forall A \in \mathcal{F}\}$ est non-vide et $\mathbb{E}_\nu(X) = \max_{Q\in core(\nu)} \mathbb{E}_Q(X), \forall X \in \chi$.*

Notons que l'implication $(ii) \Rightarrow (iii)$ peut être vue comme un résultat de représentation "robuste" pour une intégrale de Choquet par rapport à une capacité concave. L'implication $(ii) \Rightarrow (i)$ correspond à la propriéte, déjà rappelée, de sous-additivité de l'intégrale de Choquet par rapport à une capacité concave.
L'observation suivante est une application immediate du théorème ci-dessus : si la famille $\mathcal{P}$ est le noyau (ou "core") d'une capacité concave, alors l'inégalité (0.2.1) devient une égalité. Nous soulignons que le cas suivant peut se présenter (cf. exemple 1. de Huber et Strassen 1973) : la famille $\mathcal{P}$ peut être strictement incluse dans le noyau de la capacité $\sup_{P\in\mathcal{P}} P(\cdot)$, et telle que $\sup_{P\in\mathcal{P}} P(\cdot)$ soit concave.

Compte tenu des remarques ci-dessus, nous pouvons voir les deux fonctionnelles $\sup_{P\in\mathcal{P}} \mathbb{E}_P(\cdot)$ et $\mathbb{E}_{\sup_{P\in\mathcal{P}}}(\cdot)$ comme deux manières d'étendre la capacité $\sup_{P\in\mathcal{P}} P(\cdot)$ à l'espace χ, ou encore, comme deux manières de définir une "espérance" sur χ à partir d'une famille (non-vide) de probabilités $\mathcal{P}$ donnée. Soulignons que "l'espérance" $\sup_{P\in\mathcal{P}} \mathbb{E}_P(\cdot)$ est sous-additive, alors que "l'espérance" $\mathbb{E}_{\sup_{P\in\mathcal{P}}}(\cdot)$ ne l'est que si la capacité $\sup_{P\in\mathcal{P}} P(\cdot)$ est concave.

0.3 Les capacités et les intégrales de Choquet comme outils de modélisation du comportement des agents économiques

Nous avons mentionné dans cette introduction que les capacités et les intégrales de Choquet ont été appliquées dans de nombreux domaines. En particulier, ces outils ont été utilisés dans les années 1980 pour la modélisation de l'attitude des agents économiques face au risque ou à l'incertitude, et pour l'élaboration de théories alternatives à la théorie "classique" de maximisation d'espérance d'utilité de Von Neumann et Morgenstern. Nous faisons un bref rappel de certaines de ces théories :

– La théorie "duale" de Yaari (1997) : Nous nous donnons un espace de probabilité $(\Omega, \mathcal{F}, P)$. Les préférences d'un agent économique sont représentées par une fonction de distorsion ψ. La satisfaction que procure un actif contingent X est évaluée par $\mathbb{E}_{\psi\circ P}(X)$, où $\mathbb{E}_{\psi\circ P}(\cdot)$ dénote l'intégrale de Choquet par rapport à la probabilité distordue $\psi \circ P$.
– La théorie de l'utilité espérée dépendant du rang (*Rank-dependent expected utility theory*) de Quiggin (1982) : Nous nous donnons un espace de probabilité $(\Omega, \mathcal{F}, P)$. Les préférences d'un agent économique sont représentées par un couple (ψ, u), où ψ est une fonction de distorsion et $u : \mathbb{R} \to \mathbb{R}$ est une fonction croissante, appelée fonction d'utilité. La satisfaction procurée par un actif contingent X est évaluée par

$\mathbb{E}_{\psi \circ P}(u(X))$.

– La théorie de l'utilité espérée au sens de Choquet (*Choquet expected utility theory*, ou, en abrégé, CEU-theory)-cf. Schmeidler (1989), Gilboa (1987), Chateauneuf (1994) : Cette théorie intervient dans des situations où les agents économiques sont face à l'incertitude (une probabilité objective n'est pas donnée et les agents ne sont pas en mesure de se donner des probabilités subjectives). D'après cette théorie, les préférences d'un agent économique sont représentées par un couple (μ, u), où μ est une capacité et $u : \mathbb{R} \to \mathbb{R}$ est une fonction croissante. La satisfaction procurée par un actif contingent X est évaluée par $\mathbb{E}_{\mu}(u(X))$, où $\mathbb{E}_{\mu}(\cdot)$ dénote l'intégrale de Choquet par rapport à la capacité μ.

De nombreux textes présentent une synthèse de ces théories et nous renvoyons le lecteur à Denuit et al. (2006), Cohen et Tallon (2000), Chateauneuf (1994), parmi d'autres, pour plus de précisions. La terminologie en français que nous utilisons dans cette section est empruntée à J.-M. Tallon.

Nous notons que, d'un point de vue mathématique, la théorie de l'utilité espérée au sens de Choquet apparaît comme une généralisation des autres théories citées, notamment de la théorie "duale", de la théorie de l'utilité espérée dépendant du rang, ainsi que de la théorie classique de Von Neumann et Morgenstern. C'est sur la théorie de l'utilité espérée au sens de Choquet que nous basons la motivation économique du travail présenté dans cette thèse de doctorat.

Plusieurs questions analogues à celles du cadre "classique" pourraient être posées dans le cadre de la théorie de l'utilité espérée au sens de Choquet, dont la question de la comparaison de variables aléatoires (interprétées, suivant les cas, comme des positions financières, des richesses, des pertes, etc.). Cette problématique est liée à la notion de relation de dominance stochastique.

0.4 Relations de dominance stochastique "classiques"

Dans la suite nous utiliserons le terme "classique" pour désigner les notions et les résultats se rapportant au cas où l'espace mesurable sous-jacent $(\Omega, \mathcal{F})$ est muni d'une mesure de probabilité. Nous renvoyons le lecteur aux monographies de Müller et Stoyan (2002), et de Shaked et Shanthikumar (2006), pour une présentation des ordres stochastiques "classiques" et pour des références. Nous rappelons les définitions suivantes :

Définition 0.4.1 *Soit X et Y deux variables aléatoires réelles sur $(\Omega, \mathcal{F})$, soit P une probabilité sur $(\Omega, \mathcal{F})$. Nous disons que la variable aléatoire X précède la variable aléatoire Y pour la relation de dominance stochastique* croissante *(ou pour l'ordre* croissant*) par* *rapport à la probabilité P si*

$$\mathbb{E}_P(u(X)) \leq \mathbb{E}_P(u(Y))$$

pour toute fonction $u : \mathbb{R} \to \mathbb{R}$ croissante,
pourvu que les intégrales existent et soient finies.

En remplaçant l'ensemble des fonctions u croissantes, par l'ensemble des fonctions croissantes convexes (resp. par l'ensemble des fonctions croissantes convexes de la forme $x \to (x-b)_+$, où le nombre $b \in \mathbb{R}$) nous obtenons la définition de l'ordre croissant convexe (resp. de l'ordre dit "stop-loss" bien connu dans la littérature actuarielle - cf., par exemple, Denuit et al. 2006).

Définition 0.4.2 *Soit X et Y deux variables aléatoires réelles sur $(\Omega, \mathcal{F})$, soit P une probabilité sur $(\Omega, \mathcal{F})$. Nous disons que la variable aléatoire X précède la variable aléatoire Y pour la relation de dominance stochastique* croissante convexe *(ou pour l'ordre* croissant convexe*) par rapport à la probabilité P si*

$$\mathbb{E}_P(u(X)) \leq \mathbb{E}_P(u(Y))$$

pour toute fonction $u : \mathbb{R} \to \mathbb{R}$ croissante convexe,
pourvu que les intégrales existent et soient finies.

Définition 0.4.3 *Soit X et Y deux variables aléatoires réelles sur $(\Omega, \mathcal{F})$, soit P une probabilité sur $(\Omega, \mathcal{F})$. Nous disons que la variable aléatoire X précède la variable aléatoire Y pour la relation de dominance stochastique* "stop-loss" *(ou pour l'ordre* "stop-loss"*) par rapport à la probabilité P si*

$$\mathbb{E}_P((X-b)_+) \leq \mathbb{E}_P((Y-b)_+),$$

pour tout $b \in \mathbb{R}$, pourvu que les intégrales (existent et) soient finies.

Ces définitions ne font intervenir que les "lois" (par rapport à la probabilité P) des variables aléatoires X et Y, et peuvent être caractérisées en termes des fonctions de répartition F_X et F_Y de X et Y, d'une part, ainsi que des fonctions quantile q_X et q_Y de X et Y, d'autre part. La généralisation de ces caractérisations au cas où l'espace sous-jacent $(\Omega, \mathcal{F})$ est muni d'une capacité μ qui n'est pas nécessairement une mesure de probabilité sera l'objet de la section 2.3 du chapitre 2, et de la sous-section 3.3.1 du chapitre 3.

0.5 Mesures du risque statiques

Nous rappelons la définition suivante :

Définition 0.5.1 *1. Une fonctionnelle $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ est appelée une mesure du risque monétaire si elle satisfait aux propriétés suivantes pour tout $X, Y \in L^\infty(\Omega, \mathcal{F}, P)$:*

(i) *(monotonie) $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$*

(ii) *(invariance par translation) $\rho(X + b) = \rho(X) + b, \forall b \in \mathbb{R}$*

2. Une mesure du risque monétaire ρ est dite sous-additive si elle satisfait à la propriété supplémentaire de

(iii) *(sous-additivité) $\rho(X + Y) \leq \rho(X) + \rho(Y)$, $\forall X, Y \in L^\infty(\Omega, \mathcal{F}, P)$.*

3. Une mesure du risque monétaire et sous-additive ρ est dite cohérente si elle satisfait, de plus, à la propriété suivante :

(iv) *(homogénéité positive) $\rho(\lambda X) = \lambda\rho(X), \forall \lambda \in \mathbb{R}_+$.*

Le terme *monétaire* fait référence à la propriété (ii) d'invariance par translation (appelée également "cash invariance")-cf. Föllmer et Schied (2004). Le définition de mesure du risque cohérente rappelée ci-dessus coïncide, à un signe près, avec celle donnée par Artzner et al. (1999). La "convention du signe" que nous adoptons dans cette thèse de doctorat est celle utilisée dans les cas où les variables aléatoires sont interprétées comme des pertes. Dans les travaux de Föllmer et Schied (2004), et Frittelli et Rosazza Gianin (2002), les hypothèses d'homogénéité positive et sous-additivité ont été remplacées par l'hypothèse plus faible de convexité de ρ.
D'autre part, dans la littérature portant sur le calcul de primes en assurance, des fonctionelles vérifiant une propriété d'*additivité comonotone* ont été considérées, notamment par Wang (1996), Wang et al. (1997), Denneberg (1990) ; des fonctionnelles comonotonement additives ont également été considérées par Föllmer et Schied (2004).
Dans le travail de Laeven (2005), Song et Yan (2006), Heyde et al. (2007), des mesures du risque monétaires positivement homogènes et *comonotonement sous-additives* ont été étudiées. Ces mesures du risque peuvent être vues comme une généralisation des mesures du risque cohérentes de Artzner et al. (1999), ainsi que comme une généralisation des mesures du risque de Wang (1996) et Denneberg (1990).
Par ailleurs, de nombreux auteurs se sont intéressés à des mesures du risque qui, parmi d'autres propriétés, vérifient une propriété de monotonie pour une relation de dominance stochastique "classique" donnée (la terminologie mesures du risque *consistantes* est également utilisée). Parmi les travaux de ce domaine nous pouvons citer ceux de Dana (2005), Denuit et al. (2006), Song et Yan (2009), et renvoyer également aux références données par ces auteurs. Rappelons que dans le cas où l'espace de probabilité $(\Omega, \mathcal{F}, P)$ sous-jacent est sans atomes, la propriété de monotonie (ou consistance) pour l'ordre *croissant* "classique" d'une fonctionnelle ρ définie sur $L^\infty(\Omega, \mathcal{F}, P)$ est équivalente à la propriété d'invariance

en loi de ρ. Ainsi, dans la liste des auteurs déjà cités, pouvons-nous rajouter ceux qui se sont intéressés aux mesures du risque invariantes en loi- cf. Kusuoka (2001), Jouini et al. (2006), etc.
Une partie importante de cette thèse de doctorat est consacrée à l'étude de mesures du risque comonotonement additives et consistantes pour une relation de dominance stochastique "généralisée" donnée.

Les résultats de cette thèse de doctorat sont organisés en quatre chapitres. La section 1.4 du chapitre 1 et le chapitre 2 sont basés sur le document de travail Grigorova (2010). La section 1.4 du chapitre 1 a fait l'objet d'une note intitulée "Hardy-Littlewood's inequalities in the case of a capacity" parue dans la revue "Comptes Rendus Mathématique". Le chapitre 3 est basé sur le document de travail Grigorova (2011) ; nous y rajoutons ici quelques compléments.

0.6 Brève présentation des résultats du chapitre 1 : *Quelques outils mathématiques utiles*

La section 1.2 du chapitre 1 peut être considérée comme préparatoire. Nous y introduisons la terminologie qui sera utilisée dans la suite et nous faisons des rappels sur des notions de base dont les notions de fonction de répartition et fonction quantile d'une fonction mesurable par rapport à une capacité :

Définition 0.6.1 *Soit μ une capacité sur $(\Omega, \mathcal{F})$. Soit X une fonction mesurable sur $(\Omega, \mathcal{F})$ à valeurs dans $\bar{\mathbb{R}}$. La* fonction de répartition *$G_{X,\mu}$ de X par rapport à μ est définie par $G_{X,\mu}(x) := 1 - \mu(X > x)$, pour tout $x \in \bar{\mathbb{R}}$.*
Nous appelons fonction quantile *de X par rapport à μ toute inverse généralisée $r_{X,\mu}$: $(0, 1) \longrightarrow \bar{\mathbb{R}}$ de la fonction* croissante *$G_{X,\mu}$.*

Remarque 0.6.1 Le lecteur pourrait se poser la question des liens entre la fonction $G_{X,\mu}$ de la définition 0.6.1 et la fonction, soit $F_{X,\mu}$, définie par $F_{X,\mu}(x) := \mu(X \leq x)$, pour tout $x \in \bar{\mathbb{R}}$. Les deux fonctions $G_{X,\mu}$ et $F_{X,\mu}$ ne sont pas nécessairement égales. Nous remarquons le lien suivant entre les deux fonctions : $F_{X,\mu}(x) := \mu(X \leq x) = 1 - \bar{\mu}(X > x) = G_{X,\bar{\mu}}(x)$, pour tout $x \in \bar{\mathbb{R}}$. En utilisant la terminologie de la définition 0.6.1, nous avons donc que la fonction $F_{X,\mu}$ est égale à la fonction de répartition de X par rapport à la capacité duale $\bar{\mu}$.

Dans ce premier chapitre nous donnons également quelques compléments sur les fonctions quantiles par rapport à une capacité (cf. section 1.3). Nous y démontrons, en particulier, la proposition suivante :

Proposition 0.6.1 *Soit Z une fonction mesurable réelle sur $(\Omega, \mathcal{F})$, soit μ une capacité sur $(\Omega, \mathcal{F})$ et soit f une fonction* décroissante. *Supposons que f et $G_{Z,\bar{\mu}}$ n'ont pas de discontinuités en commun (où $\bar{\mu}$ dénote la capacité duale de la capacité μ). Alors, la fonction $f \circ r_{Z,\bar{\mu}}(1-\cdot)$ est une fonction quantile de $f(Z)$ par rapport à μ. En particulier,*

$$r_{f(Z),\mu}(t) = f(r_{Z,\bar{\mu}}(1-t)), \quad \textit{pour presque tout } t \in (0,1).$$

Nous y établissons également (cf. section 1.4) une généralisation des inégalités de Hardy-Littlewood dans le cas où l'espace mesurable sous-jacent est muni d'une capacité μ qui n'est pas nécessairement une probabilité. Plus précisément, nous établissons le résultat suivant :

Théorème 0.6.1 (Inegalités de Hardy-Littlewood dans le cas d'une capacité) *Soit μ une capacité sur $(\Omega, \mathcal{F})$. Soit X et Y deux fonctions mesurables positives dont les fonctions quantiles (par rapport à la capacité μ) sont dénotées $r_{X,\mu}$ et $r_{Y,\mu}$.*

1. *Si μ est concave et continue par en-dessous, alors $\mathbb{E}_\mu(XY) \leq \int_0^1 r_{X,\mu}(t) r_{Y,\mu}(t)dt$.*
2. *Si μ est convexe et continue par en-dessous, alors $\mathbb{E}_\mu(XY) \geq \int_0^1 r_{X,\mu}(1-t) r_{Y,\mu}(t)dt$.*

Nous faisons quelques remarques sur les bornes dans le théorème précédent. Nous remarquons que, comme dans le cas "classique" où μ est une probabilité, la borne supérieure dans le théorème 0.6.1 est atteinte par tout couple de fonctions mesurables comonotones. Nous remarquons que la borne inférieure est, elle aussi, atteinte. De plus, nous établissons le résultat suivant concernant la borne inférieure :

Proposition 0.6.2 *Soit μ une capacité sur $(\Omega, \mathcal{F})$ convexe et continue par en-dessous. Si $\mathbb{E}_\mu(XY) = \int_0^1 r_{X,\mu}(t) r_{Y,\mu}(1-t)dt$, pour tout couple (X, Y) de fonctions mesurables positives anti-comonotones, alors μ est une mesure de probabilité.*

0.7 Brève présentation des résultats du chapitre 2 : *Dominance stochastique par rapport à une capacité et une application à un problème d'optimisation en finance*

Nous nous intéressons d'abord à la généralisation de la notion de dominance stochastique *croissante convexe* (rappelée dans la définition 0.4.2) au cas où l'espace mesurable sous-jacent $(\Omega, \mathcal{F})$ est muni d'une capacité μ qui n'est pas nécessairement une probabilité.

Définition 0.7.1 *Soit X et Y deux fonctions mesurables sur $(\Omega, \mathcal{F})$ à valeurs réelles, et soit μ une capacité sur $(\Omega, \mathcal{F})$. Nous disons que X précède Y pour la dominance stochas-*

tique croissante convexe (par rapport à la capacité μ), noté par $X \leq_{\mathrm{icx},\mu} Y$, si

$$\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$$

pour toute fonction $u : \mathbb{R} \to \mathbb{R}$ croissante convexe,
pourvu que les intégrales de Choquet soient bien définies et finies.

Nous interprétons cette relation de deux manières : en termes de préférence "uniforme" d'un ensemble de maximiseurs d'espérance d'utilité au sens de Choquet (cf. remarque 2.3.1), ainsi qu'en termes d'ambiguïté (cf. remarque 2.3.3).
Nous établissons les résultats suivants qui généralisent des résultats bien connus dans le cas "classique" où μ est une probabilité.

Proposition 0.7.1 *Soit μ une capacité.*

(i) *Si $X \leq_{\mathrm{icx},\mu} Y$, alors $\mathbb{E}_\mu((X-b)_+) \leq \mathbb{E}_\mu((Y-b)_+)$, $\forall b \in \mathbb{R}$, pourvu que les intégrales de Choquet soient finies.*

(ii) *Si la capacité μ est continue par en-dessous et par en-dessus, alors nous avons l'implication inverse :*
si $\mathbb{E}_\mu((X-b)_+) \leq \mathbb{E}_\mu((Y-b)_+)$, $\forall b \in \mathbb{R}$, pourvu que les intégrales de Choquet soient finies, alors $X \leq_{\mathrm{icx},\mu} Y$.

Proposition 0.7.2 *Soit μ une capacité. Soit X et Y deux fonctions mesurables à valeurs réelles telles que $\int_0^1 |r_X(t)|dt < +\infty$ et $\int_0^1 |r_Y(t)|dt < +\infty$, où r_X (resp. r_Y) dénote une (version de la) fonction quantile de X (resp. de Y) par rapport à μ. Les assertions suivantes sont équivalentes :*

(i) $\mathbb{E}_\mu((X-b)_+) \leq \mathbb{E}_\mu((Y-b)_+)$, $\forall b \in \mathbb{R}$.

(ii) $\int_x^{+\infty} \mu(X > u)du \leq \int_x^{+\infty} \mu(Y > u)du$, $\forall x \in \mathbb{R}$.

(iii) $\int_y^1 r_X(t)dt \leq \int_y^1 r_Y(t)dt, \forall y \in [0,1]$.

Si, de plus, les fonctions mesurables X et Y sont bornées, alors chacune des assertions précédentes est équivalente à l'assertion

(iv) $\int_0^1 g(t)r_X(t)dt \leq \int_0^1 g(t)r_Y(t)dt$, *pour toute fonction $g : [0,1] \to \mathbb{R}_+$ croissante intégrable.*

Nous nous intéressons ensuite au problème d'optimisation suivant :

$$\begin{gathered} \text{Maximiser } \mathbb{E}_\mu(ZC) \\ \text{sous les contraintes } C \in \chi_+,\, C \leq_{icx,\mu} X \end{gathered} \tag{D}$$

où χ_+ dénote l'ensemble des fonctions mesurables positives bornées, et Z est une fonction mesurable positive telle que $\int_0^1 r_Z(t)dt < +\infty$.

Le problème est inspiré des travaux de Dybvig (1987), Jouini et Kallal (2001), Dana (2005), Föllmer et Schied (2004).

A l'aide de la version des inégalités de Hardy-Littlewood établie dans le chapitre précédent, nous démontrons le théorème :

Théorème 0.7.1 *Soit μ une capacité concave et continue par en-dessous. Pour toute fonction $X \in \chi_+$, et pour toute fonction mesurable positive Z telle que $\int_0^1 r_Z(t)dt < +\infty$ et dont la fonction de répartition G_Z par rapport à μ est continue, le problème* (D) *admet une solution et sa fonction valeur $e(X,Z)$ est donnée par : $e(X,Z) = \int_0^1 r_Z(t) r_X(t)dt$.*

Nous remarquons que l'hypothèse de continuité sur G_Z faite dans le théorème 0.7.1 peut être relaxée si la capacité μ vérifie, en plus des propriétés requises dans le théorème 0.7.1, la propriété de continuité par en-dessus.
Nous remarquons également que dans le cas où Z est "normalisée" (i.e. $\int_0^1 r_Z(t)dt = 1$) la fonction valeur du problème (D) peut être écrite de la manière suivante : $e(X,Z) = \mathbb{E}_{\psi^Z \circ \mu}(X)$, où ψ^Z est la fonction de distorsion concave définie par $\psi^Z(x) := \int_{1-x}^1 r_Z(t)dt, \forall x \in [0,1]$. Ainsi, pouvons-nous voir la fonction valeur du problème (D), à Z fixée, comme un exemple de ce que nous appellerons une mesure du risque de distorsion généralisée. Nous remarquons également que la fonctionnelle $e(\cdot, Z)$ est consistante pour la relation de dominance stochastique croissante convexe par rapport à μ.

0.8 Brève présentation des résultats du chapitre 3 : *Dominance stochastique par rapport à une capacité et mesures du risque*

Le chapitre 3 est consacré à une étude détaillée des mesures du risque comonotonement additives et consistantes pour une relation de dominance stochastique "généralisée" donnée. D'abord, nous lions la notion de dominance stochastique croissante par rapport à une capacité μ et les notions de fonction de répartition et fonction quantile par rapport à μ de la manière suivante :

Définition/Proposition 0.8.1 *Soit μ une capacité continue par en-dessous et par en-dessus. Soit X et Y deux fonctions mesurables réelles. Les assertions suivantes sont équivalentes :*

(i) $\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$, *pour toute fonction croissante $u : \mathbb{R} \to \mathbb{R}$, pourvu que les intégrales de Choquet existent et soient finies.*
Nous dirons que X précède Y pour la relation de dominance stochastique croissante par rapport à μ et nous noterons $X \leq_{mon,\mu} Y$.

(ii) $G_{X,\mu}(x) \geq G_{Y,\mu}(x),\ \forall x \in \mathbb{R}$.

(iii) $r^+_{X,\mu}(t) \leq r^+_{Y,\mu}(t), \forall t \in (0,1)$. [1]

Nous démontrons les deux résultats suivants :

Théorème 0.8.1 *Soit μ une capacité sur $(\Omega, \mathcal{F})$ continue par en-dessous et par en-dessus. La fonctionnelle $\rho : \chi \to \mathbb{R}$ est une mesure du risque monétaire satisfaisant les propriétés de*

(i) *(consistance pour la relation $\leq_{\text{mon},\mu}$) $X \leq_{\text{mon},\mu} Y \Rightarrow \rho(X) \leq \rho(Y)$ et*

(ii) *(additivité comonotone) X, Y comonotones $\Rightarrow \rho(X+Y) = \rho(X) + \rho(Y)$*

si, et seulement si, il existe une fonction de distorsion ψ telle que $\rho(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi$.

Théorème 0.8.2 *Soit μ une capacité sur $(\Omega, \mathcal{F})$. Nous supposons qu'il existe une fonction mesurable réelle Z dont la fonction de répartition $G_{Z,\mu}$ est continue et vérifie la propriété :*

$$\lim_{x\to-\infty} G_{Z,\mu}(x) = 0 \quad et \quad \lim_{x\to+\infty} G_{Z,\mu}(x) = 1.$$

La fonctionnelle $\rho : \chi \to \mathbb{R}$ est une mesure du risque monétaire vérifiant les propriétés d'additivité comonotone et consistance pour la relation de dominance stochastique "stop-loss" par rapport à μ si, et seulement si, il existe une fonction de distorsion concave *ψ telle que $\rho(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi$.*

Nous remarquons que, dans le cas particulier où μ est une probabilité, les hypothèses sur l'espace $(\Omega, \mathcal{F}, \mu)$ du théorème précédent se réduisent à l'hypothèse usuelle de non-atomicité de l'espace de probabilité sous-jacent. Nous renvoyons le lecteur à la sous-section 3.3.3 pour plus de détails sur les hypothèses du théorème précédent.
Nous obtenons également une autre manière de décrire l'ensemble de mesures du risque comonotonement additives et consistantes pour la relation de dominance stochastique "stop-loss" par rapport à μ (notée $\leq_{sl,\mu}$) :

Théorème 0.8.3 (caractérisation "de type Kusuoka" dans le cas d'une capacité) *Soit μ une capacité. Nous supposons qu'il existe une fonction mesurable réelle Z dont la fonction de répartition $G_{Z,\mu}$ est continue et vérifie la propriété :*

$$\lim_{x\to-\infty} G_{Z,\mu}(x) = 0 \quad et \quad \lim_{x\to+\infty} G_{Z,\mu}(x) = 1.$$

Soit $\rho : \chi \longrightarrow \mathbb{R}$ une fonctionnelle. Les assertions suivantes sont équivalentes :

(i) *La fonctionnelle ρ est une mesure du risque monétaire vérifiant les propriétés d'additivité comonotone et consistance pour la relation $\leq_{\text{sl},\mu}$.*

1. Le symbole $r^+_{X,\mu}$ (resp. $r^+_{Y,\mu}$) dénote la fonction quantile supérieure de X (resp. Y) par rapport à μ.

(ii) *Il existe* $\alpha \in [0,1]$ *et une fonction mesurable positive* Y *vérifiant* $\int_0^1 r_{Y,\mu}(t)dt = 1$ *telles que*

$$\rho(X) = \alpha \sup_{t<1} r^+_{X,\mu}(t) + (1-\alpha) \int_0^1 r_{Y,\mu}(t) r_{X,\mu}(t)dt, \quad \forall X \in \chi.$$

Nous lions le théorème ci-dessus à la fonction valeur du problème d'optimisation (problème D) étudié dans le chapitre précédent. Plus précisément, nous établissons le résultat suivant :

Théorème 0.8.4 *Soit* μ *une capacité* concave, *continue par en-dessous et par en-dessus. Nous supposons qu'il existe une fonction mesurable réelle* Z *dont la fonction de répartition* $G_{Z,\mu}$ *est continue. Soit* $\rho : \chi_+ \longrightarrow \mathbb{R}$ *une fonctionnelle. Les assertions suivantes sont équivalentes :*

(i) *La fonctionnelle* ρ *est une mesure du risque monétaire sur* χ_+ *vérifiant les propriétés d'additivité comonotone et consistance pour la relation* $\leq_{\mathrm{icx},\mu}$.

(ii) *Il existe* $\alpha \in [0,1]$ *et une fonction mesurable positive* Y *vérifiant* $\int_0^1 r_{Y,\mu}(t)dt = 1$ *telle que*

$$\rho(X) = \alpha \sup_{t<1} r^+_{X,\mu}(t) + (1-\alpha)\, \rho_Y(X), \quad \forall X \in \chi_+,$$

où $\rho_Y(X) := e(X,Y) = \sup_{C \in \chi_+ : C \leq_{\mathrm{icx},\mu} X} \mathbb{E}_\mu(YC)$, $\forall X \in \chi_+$.

Nous remarquons également que quelques résultats bien connus dans le cas où l'espace mesurable sous-jacent $(\Omega, \mathcal{F})$ est muni d'une probabilité ne sont pas nécessairement vrais dans le cas plus général d'une capacité (cf. sous-section 3.3.4, ainsi que les remarques 3.4.1 et 3.4.2). Nous donnons des exemples de mesures du risque pouvant être représentées comme des intégrales de Choquet par rapport à une fonction d'ensembles de la forme $\psi \circ \mu$ où ψ est une fonction de distorsion et μ est la capacité initiale. Nos exemples généralisent les notions "classiques" de Valeur au Risque (*Value at Risk*) et *Tail Value at Risk.* Dans le cas de la Valeur au Risque "généralisée" deux cas particuliers de capacité initiale μ sont considérés et des interprétations économiques sont données.

0.9 Brève présentation des résultats du chapitre 4 : *Représentation "robuste" des mesures du risque comonotonement sous-additives ou comonotonement convexes et consistantes pour une relation de dominance stochastique "généralisée" donnée*

Dans ce chapitre nous nous intéressons à des mesures du risque qui sont comonotonement sous-additives, ou comonotonement convexes, et consistantes pour la relation de dominance stochastique croissante "généralisée". L'étude de ce type de mesures du risque est inspirée des travaux de Laeven (2005), Song et Yan (2006), Heyde et al. (2007), et plus particulièrement encore, des travaux de Song et Yan (2009). Nous établissons les deux théorèmes suivants qui sont les analogues des théorèmes 3.1 et 3.5 de Song et Yan (2009) dans notre cadre.

Théorème 0.9.1 *Soit* μ *une capacité sur* $(\Omega, \mathcal{F})$ *continue par en-dessous et par en-dessus. Soit* $\rho : \chi \to \mathbb{R}$ *une mesure du risque monétaire vérifiant les propriétés de sous-additivité comonotone, homogénéité positive et consistance pour la relation* $\leq_{\text{mon},\mu}$ *de dominance stochastique croissante généralisée. La fonctionnelle* ρ *a la représentation suivante :*

$$\rho(X) = \max_{\psi \in \mathcal{D}^\rho} \mathbb{E}_{\psi \circ \mu}(X), \text{ pour tout } X \in \chi,$$

où $\mathcal{D}^\rho := \{\psi \text{ fonction de distorsion telle que } \mathbb{E}_{\psi \circ \mu}(Y) \leq \rho(Y), \text{ pour tout } Y \in \chi\}$.

Théorème 0.9.2 *Soit* μ *une capacité sur* $(\Omega, \mathcal{F})$ *continue par en-dessous et par en-dessus. Soit* $\rho : \chi \to \mathbb{R}$ *une mesure du risque monétaire vérifiant les propriétés de convexité comonotone et consistance pour la relation* $\leq_{\text{mon},\mu}$ *de dominance stochastique croissante généralisée. La fonctionnelle* ρ *a la représentation suivante :*

$$\rho(X) = \max_{\psi \in \mathcal{D}} \left(\mathbb{E}_{\psi \circ \mu}(X) - \alpha(\psi)\right), \text{ pour tout } X \in \chi,$$

où $\mathcal{D}$ *dénote l'ensemble des fonctions de distorsion, et* $\alpha(\cdot)$ *est une fonction de pénalité définie par* $\alpha(\psi) := \sup_{\{Y \in \chi : \rho(Y) \leq 0\}} \mathbb{E}_{\psi \circ \mu}(Y)$, *pour tout* $\psi \in \mathcal{D}$.

Nous mentionnons l'étude de mesures du risque comonotonement sous-additives, ou comonotonement convexes, et consistantes pour la relation de dominance stochastique *croissante convexe* "généralisée" comme un possible travail de recherche à venir.

CHAPITRE 1

Some useful mathematical tools

1.1 Introduction

This first chapter of the thesis could be seen as preparatory : it is dedicated to some useful mathematical tools. An emphasis is placed on tools in which the quantile functions with respect to a capacity intervene. The chapter contains some recalls, as well as some new mathematical results which will be useful in the sequel.
The remainder of the chapter is organized as follows. In section 1.2 we have gathered some well-known results about capacities, quantile functions with respect to a capacity and Choquet integrals. Section 1.3 contains some complements on quantile functions with respect to a capacity. The results given in that subsection are generalizations of results well-known in the particular case of a probability measure. Section 1.4 is devoted to a "generalization" of Hardy-Littlewood's inequalities to the case where the underlying measurable space is endowed with a capacity which is not necessarily a probability measure. The section is divided in two subsections : subsection 1.4.1 contains the statement of the "generalized" inequalities (cf. theorem 1.4.1), its proof and two remarks ; subsection 1.4.2 is dedicated to some comments on the lower bound of the "generalized" inequalities.

Subsection 1.4.1 of this chapter has given rise to a note published in the journal "Comptes Rendus Mathématique" under the title "Hardy-Littlewood's inequalities in the case of a capacity" (cf. Grigorova 2013). Subsection 1.4.2 of the present chapter could be seen as complementary to that note.

1.2 Some definitions and basic properties

The definitions and results recalled in this section can be found in the book by Denneberg (1994), and/or in that by Föllmer and Schied (2004) (section 4.7).
Let $(\Omega, \mathcal{F})$ be a measurable space.

Definition 1.2.1 *A set function $\mu : \mathcal{F} \longrightarrow [0,1]$ is called a capacity if it satisfies $\mu(\emptyset) = 0$ (groundedness), $\mu(\Omega) = 1$ (normalization) and the following monotonicity property : $A, B \in \mathcal{F}, A \subset B \Rightarrow \mu(A) \leq \mu(B)$.*

Definition 1.2.2 *A capacity μ is called* concave *(or submodular, or 2-alternating) if*

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \ \text{for all } A, B \in \mathcal{F}.$$

A capacity μ is called convex *(or supermodular) if it satisfies the previous property where the inequality is reversed.*

A capacity μ is called continuous from below *if*

$$(A_n) \subset \mathcal{F} \text{ such that } A_n \subset A_{n+1}, \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\cup_{n=1}^{\infty} A_n).$$

A capacity μ is called continuous from above *if*

$$(A_n) \subset \mathcal{F} \text{ such that } A_n \supset A_{n+1}, \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n).$$

The dual *capacity $\bar{\mu}$ of a given capacity μ is defined by*

$$\bar{\mu}(A) := 1 - \mu(A^c), \ \text{for all } A \in \mathcal{F}.$$

Definition 1.2.3 *Two real-valued measurable functions X and Y on $(\Omega, \mathcal{F})$ are called comonotonic if*

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0, \ \forall(\omega, \omega') \in \Omega \times \Omega.$$

The following characterization of comonotonic functions corresponds to proposition 4.5 in Denneberg (1994) (see also Föllmer and Schied 2004) :

Proposition 1.2.1 *For two real-valued measurable functions X, Y on $(\Omega, \mathcal{F})$ the following conditions are equivalent :*

(i) *X and Y are comonotonic.*

(ii) *There exists a measurable function Z on $(\Omega, \mathcal{F})$ and two non-decreasing functions f and g on $\mathbb{R}$ such that $X = f(Z)$ and $Y = g(Z)$.*

(iii) *There exist two continuous, non-decreasing functions u and v on $\mathbb{R}$ such that $u(z) + v(z) = z$, $z \in \mathbb{R}$, and $X = u(X+Y)$, $Y = v(X+Y)$.*

For a measurable function X on $(\Omega, \mathcal{F})$, the *Choquet integral* of X with respect to a capacity μ is defined as follows :

$$\mathbb{E}_\mu(X) := \int_0^{+\infty} \mu(X > x)dx + \int_{-\infty}^0 (\mu(X > x) - 1)dx.$$

Note that the Choquet integral in the preceding definition may not exist (namely, if one of the two (Riemann) integrals on the right-hand side is equal to $+\infty$ and the other to

$-\infty$), may be in $\mathbb{R}$ or may be equal to $+\infty$ or $-\infty$. The Choquet integral always exists if the function X is bounded from below or from above. The Choquet integral exists and is finite if X is bounded.

For reader's convenience and in order to fix the terminology, we summarize some of the main properties of Choquet integrals in the following propositions (cf. proposition 5.1 in Denneberg 1994), where we make the convention that the properties are valid provided the expressions make sense (which is always the case when we restrain ourselves to bounded measurable functions).

Proposition 1.2.2 *Let μ be a capacity on $(\Omega, \mathcal{F})$, and let X and Y be measurable functions on $(\Omega, \mathcal{F})$. The following properties hold true :*

- *(positive homogeneity)* $\mathbb{E}_\mu(\lambda X) = \lambda \mathbb{E}_\mu(X), \forall \lambda \in \mathbb{R}_+$
- *(monotonicity)* $X \leq Y \Rightarrow \mathbb{E}_\mu(X) \leq \mathbb{E}_\mu(Y)$
- *(translation invariance)* $\mathbb{E}_\mu(X + b) = \mathbb{E}_\mu(X) + b, \forall b \in \mathbb{R}$
- *(asymmetry)* $\mathbb{E}_\mu(-X) = -\mathbb{E}_{\bar{\mu}}(X)$, *where $\bar{\mu}$ is the dual capacity of μ ($\bar{\mu}(A)$ is defined by $\bar{\mu}(A) = 1 - \mu(A^c), \forall A \in \mathcal{F}$)*
- *(comonotonic additivity) If X and Y are (real-valued) comonotonic functions, then* $\mathbb{E}_\mu(X + Y) = \mathbb{E}_\mu(X) + \mathbb{E}_\mu(Y)$.

We recall the subadditivity property of the Choquet integral with respect to a concave capacity.

Proposition 1.2.3 *Let μ be a concave capacity on $(\Omega, \mathcal{F})$, and let X and Y be measurable real-valued functions on $(\Omega, \mathcal{F})$ such that $\mathbb{E}_\mu(X) > -\infty$ and $\mathbb{E}_\mu(Y) > -\infty$. We have the following property :*

$$\textit{(sub-additivity)} \qquad \mathbb{E}_\mu(X + Y) \leq \mathbb{E}_\mu(X) + \mathbb{E}_\mu(Y).$$

We refer the reader to Denneberg (1994) for a slightly weaker assumption than the one given in the previous proposition.

Remark 1.2.1 The reader should not be misled by the vocabulary used in the paper. We emphasize that when the capacity μ is concave in the sense of definition 1.2.2, the functional $\mathbb{E}_\mu(.)$ is a convex functional (in the usual sense) on the space of bounded real-valued measurable functions.

The reader is referred to Denneberg (1994) for the following result.

Theorem 1.2.1 (monotone convergence) *Let μ be a capacity on $(\Omega, \mathcal{F})$ which is continuous from below. For a non-decreasing sequence (X_n) of non-negative measurable functions, we have*
$\lim_{n\to\infty} \mathbb{E}_\mu(X_n) = \mathbb{E}_\mu(\lim_{n\to\infty} X_n)$.

We recall the notions of (non-decreasing) distribution function and of a quantile function with respect to a capacity μ (cf. Föllmer and Schied 2004).

Definition 1.2.4 *Let X be a measurable function on $(\Omega, \mathcal{F})$. The distribution function G_X of X with respect to μ is defined by $G_X(x) := 1 - \mu(X > x)$, for all $x \in \bar{\mathbb{R}}$.*

The non-decreasingness of G_X is due to the monotonicity of μ.
In the case where μ is a probability measure, the distribution function G_X coincides with the usual (cumulative) distribution function F_X of X defined by $F_X(x) := \mu(X \leq x)$, $\forall x \in \bar{\mathbb{R}}$.

Let us now recall the notion of a generalized inverse of the (non-decreasing) function G_X.

Definition 1.2.5 *For a measurable function X defined on $(\Omega, \mathcal{F})$ and for a capacity μ, let G_X denote the distribution function of X with respect to μ. We call a quantile function of X with respect to μ every function $r_X : (0,1) \longrightarrow \bar{\mathbb{R}}$ verifying*

$$\sup\{x \in \mathbb{R} \mid G_X(x) < t\} \leq r_X(t) \leq \sup\{x \in \mathbb{R} \mid G_X(x) \leq t\}, \ \forall t \in (0,1),$$

where the convention $\sup\{\varnothing\} = -\infty$ is used.
The functions r_X^- and r_X^+ defined by

$$r_X^-(t) := \sup\{x \in \mathbb{R} \mid G_X(x) < t\}, \forall t \in (0,1) \ \text{ and } \ r_X^+(t) := \sup\{x \in \mathbb{R} \mid G_X(x) \leq t\}, \forall t \in (0,1)$$

are called the lower and upper quantile functions of X with respect to μ.

For notational convenience, we omit the dependence on μ in the notation G_X and r_X when there is no ambiguity.
The following observation can be found in Föllmer and Schied (2004).

Remark 1.2.2 The lower and upper quantile functions of X with respect to μ can be expressed in the following manner as well :

$$r_X^-(t) := \inf\{x \in \mathbb{R} \mid G_X(x) \geq t\}, \forall t \in (0,1) \ \text{ and } \ r_X^+(t) := \inf\{x \in \mathbb{R} \mid G_X(x) > t\}, \forall t \in (0,1)$$

In the following well-known result we make the convention that the assertion is valid provided the expressions make sense. The result can be found in Denneberg (1994) (cf. pages 61-62 in chapter 5), or in Yan (2009) (subsection 5.1), as well as in Föllmer and Schied (2004) for the bounded case.

Proposition 1.2.4 *Let X be a measurable function and let r_X be a quantile function of X with respect to a capacity μ, then $\mathbb{E}_\mu(X) = \int_0^1 r_X(t)dt$.*

The following remark is due to the monotonicity of the function r_X.

Remark 1.2.3 The function r_X being *non-decreasing* on $(0,1)$, the Lebesgue integral $\int_{(0,1)} r_X d\lambda$ makes sense if and only if the generalized Riemann integral $\int_0^1 r_X(t)dt$ makes sense. Moreover, the function r_X is integrable in the Lebesgue sense if and only if its generalized Riemann integral exists and is finite. A presentation of the generalized Riemann integral of an extended real-valued monotonic function can be found in the book of Denneberg (1994).

The following lemma is the analogue in the case of a capacity of lemma A.23. in Föllmer and Schied (2004), and can be found in Denneberg (1994) (cf. also proposition 3.2 in Yan 2009).

Lemma 1.2.1 *Let $X = f(Y)$ where f is a non-decreasing function and let r_Y be a quantile function of Y with respect to a capacity μ. Suppose that f and G_Y have no common discontinuities, then $f \circ r_Y$ is a quantile function of X with respect to μ. In particular,*

$$r_X(t) = r_{f(Y)}(t) = f(r_Y(t)) \text{ for almost every } t \in (0,1),$$

where r_X denotes a quantile function of X with respect to μ.

Remark 1.2.4 If the capacity μ satisfies the additional properties of continuity from below and from above, the assumption of no common discontinuities of the functions f and G_Y can be dropped in the previous lemma. The proof is then analogous to the proof in the classical case of a probability measure (cf. lemma A.23. in Föllmer and Schied 2004 for a proof in the classical case) and is left to the reader.

The following property is an immediate consequence of the above lemma 1.2.1.

Property 1.2.1 *If $\lambda \geq 0$, then $r_{\lambda X}(t) = \lambda r_X(t)$, for almost every $t \in (0,1)$.*

The notion of comonotonic functions proves to be very useful while dealing with Choquet integrals thanks to the following result (cf. lemma 4.84 in Föllmer and Schied 2004, as well as corollary 4.6 in Denneberg 1994).

Lemma 1.2.2 *If $X, Y : \Omega \to \mathbb{R}$ is a pair of comonotonic measurable functions and if r_X, r_Y, r_{X+Y} are quantile functions (with respect to a capacity μ) of $X, Y, X+Y$ respectively, then*

$$r_{X+Y}(t) = r_X(t) + r_Y(t), \text{ for almost every } t.$$

1.3 Some complements on quantile functions with respect to a capacity

We state a useful result about monotonic transformations of measurable functions and the corresponding upper quantile functions.

Lemma 1.3.1 *Let Z be a real-valued measurable function on $(\Omega, \mathcal{F})$, let μ be a capacity on $(\Omega, \mathcal{F})$ and let f be a non-decreasing right-continuous function. Denote by r_Z^+ and by $r_{f(Z)}^+$ the upper quantile functions of Z and $f(Z)$ (with respect to μ). Suppose that f and G_Z have no common discontinuities, then*

$$r_{f(Z)}^+(t) = f(r_Z^+(t)),\ \forall t \in (0,1).$$

Proof : The proof of the lemma uses arguments similar to those used in the proof of proposition 3.2 in Yan (2009) and is given in the appendix.

□

An analogous result to that of lemma 1.3.1 holds true in the case of lower quantile functions with respect to a capacity. The result can be shown by using similar arguments to the ones used in the proof of the previous lemma 1.3.1 - its proof is therefore omitted.

Lemma 1.3.2 *Let Z be a real-valued measurable function on $(\Omega, \mathcal{F})$, let μ be a capacity on $(\Omega, \mathcal{F})$ and let f be a non-decreasing* left-continuous *function. Denote by r_Z^- and by $r_{f(Z)}^-$ the* lower *quantile functions of Z and $f(Z)$ (with respect to μ). Suppose that f and G_Z have no common discontinuities, then*

$$r_{f(Z)}^-(t) = f(r_Z^-(t)),\ \forall t \in (0,1).$$

Using the previous two lemmas 1.3.1 and 1.3.2, we state a proposition representing a generalization to the case of a capacity of a well-known "classical" result about the upper and lower quantile functions of comonotonic random variables - cf. for instance theorem 4.2.1 in Dhaene et al. (2006) for the classical case.

Proposition 1.3.1 *If X and Y are two comonotonic real-valued measurable functions, then*

$$r_{X+Y}^+(t) = r_X^+(t) + r_Y^+(t),\ \forall t \in (0,1) \qquad \text{and} \tag{1.3.1}$$

$$r_{X+Y}^-(t) = r_X^-(t) + r_Y^-(t),\ \forall t \in (0,1). \tag{1.3.2}$$

Proof: The arguments of the proof of proposition 1.3.1 are similar to those used in the proof of corollary 4.6 in Denneberg (1994). The proof is placed in the appendix. □

Remark 1.3.1 The previous proposition 1.3.1 is to be compared with lemma 1.2.2. In fact, lemma 1.2.2 can be viewed as a consequence of proposition 1.3.1 after recalling that a quantile function (with respect to a given capacity) of a given real-valued measurable function is unique except on an at most countable set.

In order to complete the presentation we provide some results on the quantile functions (with respect to a capacity μ) of *non-increasing* transformations of measurable functions. The following proposition can be seen as an analogue of lemma 1.2.1 in the case where the (transformation) function f is non-increasing. The proposition is well-known in the case where the capacity μ is a probability measure (cf., for instance, Föllmer and Schied 2004, lemma A.23).

Proposition 1.3.2 *Let Z be a real-valued measurable function on $(\Omega, \mathcal{F})$, let μ be a capacity on $(\Omega, \mathcal{F})$ and let f be a* non-increasing *function. We suppose that f and $G_{Z,\bar{\mu}}$ have no common discontinuities (where $\bar{\mu}$ denotes the dual capacity of the capacity μ). Then, the function $f \circ r_{Z,\bar{\mu}}(1-\cdot)$ is a quantile function with respect to μ of $f(Z)$. In particular,*

$$r_{f(Z),\mu}(t) = f(r_{Z,\bar{\mu}}(1-t)), \text{ for almost every } t \in (0,1).$$

The proof of the previous proposition 1.3.2 is based on two lemmas which we present hereafter.
The following lemma shows that if two non-decreasing functions are equal except on an at most countable set, then their upper (resp. lower) generalized inverse functions are equal.

Lemma 1.3.3 *Let $h_1 : \mathbb{R} \longrightarrow \mathbb{R}$ and $h_2 : \mathbb{R} \longrightarrow \mathbb{R}$ be two non-decreasing functions such that*

$$h_1 = h_2 \text{ except on an at most countable set.} \tag{1.3.3}$$

The following statements hold true :

1. *The* upper *generalized inverse functions $\check{h}_1$ and $\check{h}_2$ of the functions h_1 and h_2 are equal.*
2. *The* lower *generalized inverse functions of the functions h_1 and h_2 are equal.*

Proof : Let us prove the first statement ; the second statement can be proved by means of similar arguments.
We recall that the upper generalized inverse funtions $\check{h}_1$ and $\check{h}_2$ are defined by

$$\check{h}_1(x) := \sup\{y \in \mathbb{R} : h_1(y) \leq x\}, \forall x \in \mathbb{R} \;\text{ and }\; \check{h}_2(x) := \sup\{y \in \mathbb{R} : h_2(y) \leq x\}, \forall x \in \mathbb{R}.$$

We consider two cases.

- If $x \in \mathbb{R}$ is such that the set $\{y : h_1(y) \leq x\}$ is empty, then the set $\{y : h_2(y) \leq x\}$ is empty (otherwise, there would be a contradiction with the assumption (1.3.3)). Hence, the equality $\check{h}_1(x) = \check{h}_2(x) = \sup\{\varnothing\} = -\infty$ holds.
- If $x \in \mathbb{R}$ is such that the set $\{y : h_1(y) \leq x\}$ is not empty, then $\check{h}_1(x)$ belongs to $\mathbb{R}\cup\{+\infty\}$. Moreover, for all $y < \check{h}_1(x)$, $h_1(y) \leq x$. The previous statement, combined with the assumption (1.3.3) and the non-decreasingness of the function h_2, implies that,
$$h_2(y) \leq x, \text{ for all } y < \check{h}_1(x). \tag{1.3.4}$$
In order to prove (1.3.4), suppose, by way of contradiction, that there esists $y_0 \in \mathbb{R}$ satisfying $y < \check{h}_1(x)$ and $h_2(y_0) > x$. The function h_2 being non-decreasing, we have $h_2(y) \geq h_2(y_0) > x$, for all $y \in (y_0, \check{h}_1(x))$. Thus, $h_2(y) > h_1(y)$, for all $y \in (y_0, \check{h}_1(x))$, which contradicts the assumption (1.3.3) on the functions h_1 and h_2. The assertion (1.3.4) is thus proved.
The inclusion $]-\infty, \check{h}_1(x)[\subset \{y : h_2(y) \leq x\}$ which we have just proved and the definition of $\check{h}_2(x)$ give $\check{h}_1(x) \leq \check{h}_2(x)$.

In both of the cases the inequality $\check{h}_1(x) \leq \check{h}_2(x)$ holds true.
Interchanging the roles of the functions h_1 and h_2 in the previous reasoning allows us to prove the converse inequality, namely $\check{h}_2(x) \leq \check{h}_1(x)$. The desired result is thus proved.

□

Remark 1.3.2 The result of the previous lemma holds true also in the case where the functions h_1 and h_2 are defined on the extended real line $\bar{\mathbb{R}}$.

Remark 1.3.3 It follows from the previous lemma (lemma 1.3.3) and from the definition of the generalized inverse of a non-decreasing function that two non-decreasing functions which are equal except on an at most countable set have the same generalized inverse functions. More precisely, we have the following observation :
Let h_1 and h_2 be two functions satisfying the assumptions of the previous lemma 1.3.3. A function r is a generalized inverse of h_1 if and only if r is a generalized inverse of h_2.

Lemma 1.3.4 *Let μ be a capacity and let Z be a measurable function. The following two assertions hold true :*

1. $r^+_{-Z,\mu}(t) = -r^-_{Z,\bar{\mu}}(1-t)$, *for all* $t \in (0,1)$.
2. $r^-_{-Z,\mu}(t) = -r^+_{Z,\bar{\mu}}(1-t)$, *for all* $t \in (0,1)$.

Proof : Let us prove the first assertion. The proof is based on lemma 1.3.3. Let us consider the two functions h_1 and h_2 defined by :
$$h_1(x) := G_{-Z,\mu}(x) := 1 - \mu(-Z > x), \text{ for all } x \in \mathbb{R},$$
$$h_2(x) := 1 - \mu(-Z \geq x), \text{ for all } x \in \mathbb{R}.$$

The functions h_1 and h_2 are non-decreasing. Moreover, the functions h_1 and h_2 are equal except on an at most countable set. By applying lemma 1.3.3, we obtain that the corresponding upper quantile functions $\check{h}_1$ and $\check{h}_2$ are equal, i.e. $\check{h}_1 = \check{h}_2$. Let us compute $\check{h}_2$. For $t \in (0,1)$, we have

$$\begin{aligned} \check{h}_2(t) &= \sup\{x : 1 - \mu(-Z \geq x) \leq t\} = \sup\{x : \mu(Z \leq -x) \geq 1 - t\} \\ &= \sup\{x : 1 - \bar{\mu}(Z > -x) \geq 1 - t\} = \sup\{x : G_{Z,\bar{\mu}}(-x) \geq 1 - t\} \\ &= -\inf\{x : G_{Z,\bar{\mu}}(x) \geq 1 - t\} = -r^-_{Z,\bar{\mu}}(1 - t). \end{aligned} \tag{1.3.5}$$

Combining the previous equation (1.3.5) with the equality $\check{h}_1 = \check{h}_2$ gives :

$$\check{h}_1(t) = \check{h}_2(t) = -r^-_{Z,\bar{\mu}}(1 - t), \text{ for all } t \in (0,1).$$

On the other hand, by definition of the upper quantile function, we have $\check{h}_1(t) = r^+_{-Z,\mu}(t)$, for all $t \in (0,1)$. We conclude that $r^+_{-Z,\mu}(t) = -r^-_{Z,\bar{\mu}}(1-t)$, for all $t \in (0,1)$.
The second assertion of the lemma can be obtained by applying the first assertion with the capacity $\bar{\mu}$ and the measurable function $-Z$.

□

Remark 1.3.4 We note that the property of asymmetry of the Choquet integral (recalled in proposition 1.2.2) could be retrieved by applying proposition 1.2.4 and the previous lemma 1.3.4. We recall that the other properties of the Choquet integral from proposition 1.2.2 could also be proved by using the properties of the quantile functions with respect to a capacity (cf. Denneberg 1994).

Let us prove proposition 1.3.2.
Proof of proposition 1.3.2 : We set, for the easing of the presentation, $g := -f$. Thus,

$$r^+_{f(Z),\mu}(t) = r^+_{-g(Z),\mu}(t), \text{ for all } t \in (0,1). \tag{1.3.6}$$

By applying the previous lemma 1.3.4 with the capacity μ and the measurable function $g(Z)$, we obtain

$$r^+_{-g(Z),\mu}(t) = -r^-_{g(Z),\bar{\mu}}(1-t), \text{ for all } t \in (0,1). \tag{1.3.7}$$

Thanks to the assumptions, the function g is non-decreasing, and the functions g and $G_{Z,\bar{\mu}}$ do not have common discontinuities. By applying lemma 1.2.1 (with the non-decreasing function g and the capacity $\bar{\mu}$), we obtain

$$r^-_{g(Z),\bar{\mu}}(1-t) = g\big(r_{Z,\bar{\mu}}(1-t)\big), \text{ for almost every } t \in (0,1). \tag{1.3.8}$$

By combining equations (1.3.6), (1.3.7), and (1.3.8), and by using the definition of g, we obtain $r_{f(Z),\mu}(t) = -g\big(r_{Z,\bar{\mu}}(1-t)\big) = f\big(r_{Z,\bar{\mu}}(1-t)\big)$, for almost every $t \in (0,1)$. □

Remark 1.3.5 If the capacity μ satisfies the additional properties of continuity from below and from above, the assumption of no common discontinuities of the functions f and $G_{Z,\bar{\mu}}$ in the previous proposition 1.3.2 can be dropped. This remark is analogous to remark 1.2.4 ; its proof is omitted.

The two statements of the following result can be seen as analogues of lemmas 1.3.1 and 1.3.2 in the case of a *non-increasing* transformation of a measurable function. The proof uses arguments similar to those used in the proof of the previous proposition 1.3.2 and is given for reader's convenience.

Proposition 1.3.3 *Let μ be a capacity and let Z be a measurable function.*

1. *Let f be a* non-increasing left-continuous *function. If f and $G_{Z,\bar{\mu}}$ have no common discontinuities (where $\bar{\mu}$ denotes the dual capacity of the capacity μ), then*
$$r^+_{f(Z),\mu}(t) = f\big(r^-_{Z,\bar{\mu}}(1-t)\big), \text{ for all } t \in (0,1).$$
2. *Let f be a* non-increasing right-continuous *function. If f and $G_{Z,\bar{\mu}}$ have no common discontinuities (where $\bar{\mu}$ denotes the dual capacity of the capacity μ), then*
$$r^-_{f(Z),\mu}(t) = f\big(r^+_{Z,\bar{\mu}}(1-t)\big), \text{ for all } t \in (0,1).$$

Proof : Let us prove the first assertion. The proof is based on lemmas 1.3.4 and 1.3.2. We set, for the easing of the presentation, $g := -f$. By applying lemma 1.3.4 with the capacity μ and the measurable function $g(Z)$, we obtain

$$r^+_{f(Z),\mu}(t) = r^+_{-g(Z),\mu}(t) = -r^-_{g(Z),\bar{\mu}}(1-t), \text{ for all } t \in (0,1). \tag{1.3.9}$$

Thanks to the assumptions, the function g is non-decreasing and left-continuous, and the functions g and $G_{Z,\bar{\mu}}$ do not have common discontinuities. By applying lemma 1.3.2 (with the non-decreasing left-continuous function g and the capacity $\bar{\mu}$), we obtain

$$r^-_{g(Z),\bar{\mu}}(1-t) = g\big(r^-_{Z,\bar{\mu}}(1-t)\big), \text{ for all } t \in (0,1). \tag{1.3.10}$$

By using the previous two equations and the definition of g, we obtain the desired result, namely

$$r^+_{f(Z),\mu}(t) = f\big(r^-_{Z,\bar{\mu}}(1-t)\big), \text{ for all } t \in (0,1).$$

The proof of the second assertion is based on lemmas 1.3.4 and 1.3.1 ; it follows the same reasoning as the above, and is left to the reader.

□

Remark 1.3.6 The previous result could be seen as a generalization to the case of a capacity (which is not necessarily a probability measure) of a well-known "classical" result (cf. the second part of lemma 2.1 in Dhaene et al. 2006 for the "classical" case).

1.4 Hardy-Littlewood's inequalities in the case of a capacity

In this section we establish a useful result which can be seen as a "generalization" of the well-known Hardy-Littlewood's inequalities to the present setting.
For the statement and the proof of this result in the classical case of a probability measure we refer to theorem A. 24 in Föllmer and Schied (2004), as well as to the references therein ; some applications of the "classical" Hardy-Littlewood's inequalities to finance can be found in the same reference. Other applications of the "classical" version to economics and finance can be found in Carlier and Dana (2006) ; see also Carlier and Dana (2005) (and the references therein) where a supermodular extension of the "classical" inequalities is used in insurance.
The generalization that we state in this section will be used in solving the optimization problem of the following chapter. This generalized version proves to be also useful in our ongoing work concerning some static optimization problems related to the Choquet expected utility theory.

1.4.1 The theorem

The contents of this subsection correspond to our note "Hardy-Littlewood's inequalities in the case of a capacity"-cf. Grigorova (2013).

Theorem 1.4.1 (Hardy-Littlewood's inequalities in the case of a capacity) *Let μ be a capacity on $(\Omega, \mathcal{F})$. Let X and Y be two non-negative measurable functions with quantile functions (with respect to the capacity μ) denoted by r_X and r_Y.*

1. *If μ is submodular and continuous from below, then $\mathbb{E}_\mu(XY) \leq \int_0^1 r_X(t) r_Y(t)dt$.*
2. *If μ is supermodular and continuous from below, then $\mathbb{E}_\mu(XY) \geq \int_0^1 r_X(1-t) r_Y(t)dt$.*

The proof is based on the following lemma.

Lemma 1.4.1 *Let μ be a capacity on $(\Omega, \mathcal{F})$ which is continuous from below. Let (X_n) be a non-decreasing sequence of non-negative measurable functions and let X denote the limit function.*

1. *The sequence of distribution functions (with respect to μ) of X_n is non-increasing and converges to the distribution function (with respect to μ) of X i.e. $G_{X_n}(x) \downarrow G_X(x)$, for all $x \in \bar{\mathbb{R}}_+$.*
2. *The following convergence holds as well : $\quad r_{X_n}(t) \uparrow r_X(t)$ for almost every t, where r_{X_n} and r_X stand for (versions of) the quantile functions (with respect to μ) of X_n and X, respectively.*

Proof of the lemma : The proof of the first statement is contained in the proof of theorem 8.1 in Denneberg (1994).
To prove the second statement we will use the lower quantile function $r^-_{X_n}$ of X_n defined by :

$$r^-_{X_n}(t) := \sup\{x \in \mathbb{R} : G_{X_n}(x) < t\}, \text{ for } t \in (0,1).$$

The sequence (X_n) being non-negative, non-decreasing, we have that the sequence $(r^-_{X_n})$ is non-negative, non-decreasing and we denote by r its limit function i.e. $r(t) := \lim_n r^-_{X_n}(t) = \sup_n r^-_{X_n}(t), \forall t \in (0,1)$. We will show that for all $t \in (0,1), r(t) = r^-_X(t)$, where $r^-_X(t) := \sup\{x \in \mathbb{R} : G_X(x) < t\}$ is the lower quantile function of X (with respect to μ). The conclusion of the lemma will follow as $r^-_X = r_X$ almost everywhere and $r^-_{X_n} = r_{X_n}$ almost everywhere.
Now, $G_{X_n} \geq G_X$ for all n, which implies that $r^-_{X_n}(t) \leq r^-_X(t), \forall t \in (0,1), \forall n$. By passing to the limit, we obtain $r(t) \leq r^-_X(t), \forall t \in (0,1)$.
We turn to the proof of the converse inequality, namely $r(t) \geq r^-_X(t), \forall t \in (0,1)$. Fix $t \in (0,1)$ and let $x \in \mathbb{R}$ be such that $G_X(x) < t$. By the first part of the lemma, we know that $G_{X_n}(x) \downarrow G_X(x)$. Hence, there exists $n_0 = n_0(t,x)$ such that for all $n \geq n_0$, $G_{X_n}(x) < t$. Therefore, for all $n \geq n_0, x \in \{y \in \mathbb{R} : G_{X_n}(y) < t\}$ which implies that $r^-_{X_n}(t) := \sup\{y \in \mathbb{R} : G_{X_n}(y) < t\} \geq x, \forall n \geq n_0$. By passing to the limit, we obtain that $r(t) \geq x$, which gives the desired inequality and concludes the proof.

□

Proof of theorem 1.4.1 : We will prove the first part of the theorem which concerns the upper bound. The lower bound can be proved by means of similar arguments.
Step 1. The inequality is satisfied by X and Y of the form $X = \mathbb{I}_A$, $Y = \mathbb{I}_B$, where $A, B \in \mathcal{F}$ (even without the assumption of continuity from below and submodularity of μ). Indeed,

$$\mathbb{E}_\mu(\mathbb{I}_A\mathbb{I}_B) = \mu(A \cap B) \leq \mu(A) \wedge \mu(B) = \int_0^1 r_{\mathbb{I}_A}(t) r_{\mathbb{I}_B}(t)dt, \tag{1.4.1}$$

where we have used that $r_{\mathbb{I}_A} = \mathbb{I}_{(1-\mu(A),1]}$ a.e. in order to obtain the last equality in (1.4.1).
Step 2. We prove the desired inequality for non-negative step functions. Let X and Y be two non-negative step functions. The function X has the following representation $X = \sum_{i=1}^n x_i \mathbb{I}_{A_i}$, with $x_i \geq 0$ and $A_i \in \mathcal{F}$. Without loss of generality, we can suppose that the numbers x_i are ranged in a descending order (i.e. $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$) and that the sets A_i are disjoint. Thus, the function X can be rewritten in the following manner : $X = \sum_{i=1}^n \tilde{x}_i \mathbb{I}_{\tilde{A}_i}$, where $\tilde{x}_i := x_i - x_{i+1} \geq 0$, $x_{n+1} := 0$ and $\tilde{A}_i := \cup_{k=1}^i A_k$. We note that the functions $\tilde{x}_i \mathbb{I}_{\tilde{A}_i}$ and $\tilde{x}_j \mathbb{I}_{\tilde{A}_j}$ are comonotonic. In the same manner, the function Y has the following representation : $Y = \sum_{j=1}^m \tilde{y}_j \mathbb{I}_{\tilde{B}_j}$, where $\tilde{y}_j \geq 0$ and $\tilde{B}_j \subset \tilde{B}_{j+1}$.

Thanks to the subadditivity of the Choquet integral with respect to a submodular capacity and to the positive homogeneity of the Choquet integral, we have

$$\mathbb{E}_\mu(XY) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{x}_i \tilde{y}_j \mu(\tilde{A}_i \cap \tilde{B}_j). \tag{1.4.2}$$

On the other hand, we see that $r_X = \sum_{i=1}^{n} r_{X_i}$ a.e. where we have set $X_i := \tilde{x}_i \mathbb{I}_{\tilde{A}_i}$ and where r_{X_i} designates a quantile function of X_i. Indeed, as mentioned above, the functions in the sum $\sum_{i=1}^{n} \tilde{x}_i \mathbb{I}_{\tilde{A}_i}$ are pairwise comonotonic ; therefore, the functions $\sum_{i=1}^{k-1} \tilde{x}_i \mathbb{I}_{\tilde{A}_i}$ and $\tilde{x}_k \mathbb{I}_{\tilde{A}_k}$ are comonotonic ; lemma 1.2.2 and a reasoning by induction allow us to conclude. By the same arguments, $r_Y = \sum_{j=1}^{m} r_{Y_j}$ a.e. where $Y_j := \tilde{y}_j \mathbb{I}_{\tilde{B}_j}$ and r_{Y_j} designates a quantile function of Y_j. So,

$$\int_0^1 r_X(t) r_Y(t) dt = \sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{x}_i \tilde{y}_j \int_0^1 r_{\mathbb{I}_{\tilde{A}_i}}(t) r_{\mathbb{I}_{\tilde{B}_j}}(t) dt, \tag{1.4.3}$$

where the non-negativity of $\tilde{x}_i$ and $\tilde{y}_j$, and property 1.2.1 have been used.
From the first step of the proof about indicator functions, we know that $\mu(\tilde{A}_i \cap \tilde{B}_j) \leq \int_0^1 r_{\mathbb{I}_{\tilde{A}_i}}(t) r_{\mathbb{I}_{\tilde{B}_j}}(t) dt$ (cf. equation (1.4.1)). The second step is proved, by combining this observation with equations (1.4.2) and (1.4.3).
Step 3. To prove the inequality in the general case, let X and Y be two measurable non-negative functions. Let (X_n) be a sequence of non-negative step functions such that $X_n \uparrow X$, and let (Y_n) be a sequence of non-negative step functions such that $Y_n \uparrow Y$. From the second step of the proof, we know that $\mathbb{E}_\mu(X_n Y_n) \leq \int_0^1 r_{X_n}(t) r_{Y_n}(t) dt$, for all n. By applying the monotone convergence theorem (theorem 1.2.1) to the non-negative, non-decreasing sequence $(X_n Y_n)$, we obtain $\lim_{n\to\infty} \mathbb{E}_\mu(X_n Y_n) = \mathbb{E}_\mu(XY)$. On the other hand, by using lemma 1.4.1, we obtain $r_{X_n}(t) \uparrow r_X(t)$ for almost every t and $r_{Y_n}(t) \uparrow r_Y(t)$ for almost every t ; these considerations, along with the non-negativity of $r_{X_n}(\cdot)$ and $r_{Y_n}(\cdot)$ for all n, lead to $r_{X_n}(t) r_{Y_n}(t) \uparrow r_X(t) r_Y(t)$ for almost every t. The monotone convergence theorem for Lebesgue integrals, applied to the sequence $(r_{X_n}(\cdot) r_{Y_n}(\cdot))$, gives $\lim_{n\to\infty} \int_0^1 r_{X_n}(t) r_{Y_n}(t) dt = \int_0^1 r_X(t) r_Y(t) dt$, which concludes the proof.

$\square$

Remark 1.4.1 We note that, as in the particular case where μ is a probability measure, the upper bound in theorem 1.4.1 is attained by any pair of non-negative comonotonic measurable functions. We remark, as well, that a result analogous to theorem 1.4.1 can be established in the case where $\mu(\Omega)$ is finite, but not necessarily normalized to 1.

Remark 1.4.2 In the case where the measurable functions can take negative values, theorem 1.4.1 does not necessarily hold true, as can be seen from the following counter-example.

Let $(\Omega, \mathcal{F}, \mu)$ be given, where μ is a non-additive submodular (resp. *supermodular*) capacity. Then, there exists $A \in \mathcal{F}$ such that $\mu(A) >$ (resp. $<$) $1 - \mu(A^c)$. We set $X := \mathbb{I}_A$ and $Y := b$, where $b < 0$. An explicit computation gives $\mathbb{E}_\mu(XY) = b(1 - \mu(A^c))$ and $\int_0^1 r_X(t) r_Y(t) dt = \int_0^1 r_X(t) r_Y(1-t) dt = b\mu(A)$. Thus, $\mathbb{E}_\mu(XY) > \int_0^1 r_X(t) r_Y(t) dt$ (resp. $\mathbb{E}_\mu(XY) < \int_0^1 r_X(t) r_Y(1-t) dt$), which is a violation of the upper (resp. *lower*) bound in theorem 1.4.1.

1.4.2 Some remarks on the lower bound in the "generalized" Hardy-Littlewood's inequalities [1]

The lower bound is attained

As the upper bound, the lower bound in the "generalized" Hardy-Littlewood's inequalities (theorem 1.4.1) is attained, i.e. there exists a pair of non-negative measurable functions X and Y for which the inequality of the second statement in theorem 1.4.1 becomes an equality. The following is an example of such a pair.
Let $A \in \mathcal{F}$. We set $X := 1 - \mathbb{I}_A$ and $Y := \frac{1}{2}(1 + \mathbb{I}_A)$. An explicit computation gives :

$\mathbb{E}_\mu(XY) = \mathbb{E}_\mu\left(\frac{1}{2}\mathbb{I}_{A^c}\right) = \frac{1}{2}\mu(A^c) = \frac{1}{2}(1 - \bar{\mu}(A))$.

On the other hand, by using the properties of the quantile function of a *non-increasing* transformation of a given measurable function (cf. prop. 1.3.2), we obtain

$$r_{X,\mu}(t) = r_{1-\mathbb{I}_A,\mu}(t) = 1 - r_{\mathbb{I}_A,\bar{\mu}}(1-t) = 1 - \mathbb{I}_{(1-\bar{\mu}(A),1]}(1-t) = 1 - \mathbb{I}_{[0,\bar{\mu}(A))}(t), \quad (1.4.4)$$

where the equalities hold almost everywhere.
By using the properties of the quantile function of a *non-decreasing* transformation of a given measurable function (cf. lemma 1.2.1), we obtain

$$r_{Y,\mu}(t) = r_{\frac{1}{2}(1+\mathbb{I}_A),\mu}(t) = \frac{1}{2}(1 + r_{\mathbb{I}_A,\mu}(t)) = \frac{1}{2}(1 + \mathbb{I}_{(1-\mu(A),1]}(t)), \quad (1.4.5)$$

where the equalities hold almost everywhere.
Let us compute $\int_0^1 r_{X,\mu}(t) r_{Y,\mu}(1-t) dt$. By using equations (1.4.4) and (1.4.5), we obtain

$$\int_0^1 r_{X,\mu}(t) r_{Y,\mu}(1-t) dt = \frac{1}{2}\int_0^1 1 - \mathbb{I}_{[0,\bar{\mu}(A))}(t) + \mathbb{I}_{[0,\mu(A))}(t) - \mathbb{I}_{[0,\bar{\mu}(A))}(t)\mathbb{I}_{[0,\mu(A))}(t) dt. \quad (1.4.6)$$

We recall that under the assumption of convexity of μ (which is made in the second statement of theorem 1.4.1), $\mu(A) \leq \bar{\mu}(A)$. Therefore, $[0, \bar{\mu}(A)) \cap [0, \mu(A)) = [0, \mu(A))$. This observation, combined with equation (1.4.6), gives $\int_0^1 r_{X,\mu}(t) r_{Y,\mu}(1-t) dt = \frac{1}{2}(1 - \bar{\mu}(A))$.
We conclude that, in the case where μ is convex, $\mathbb{E}_\mu(XY) = \frac{1}{2}(1 - \bar{\mu}(A)) = \int_0^1 r_{X,\mu}(t) r_{Y,\mu}(1-t) dt$.

1. This subsection could be seen as complementary to our note "Hardy-Littlewood's inequalities in the case of a capacity" (Grigorova 2013).

The lower bound and anti-comonotonic measurable functions

Let us recall the definition of anti-comonotonic measurable functions.

Definition 1.4.1 *Two real-valued measurable functions X and Y on $(\Omega, \mathcal{F})$ are called anti-comonotonic if X and $-Y$ are comonotonic.*

For reader's convenience we also recall the following proposition, which is an immediate consequence of the definition of anti-comonotonic measurable functions (definition 1.4.1) and of proposition 1.2.1.

Proposition 1.4.1 *For two real-valued measurable functions X, Y on $(\Omega, \mathcal{F})$ the following conditions are equivalent :*

(i) *X and Y are anti-comonotonic.*

(ii) $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \leq 0, \ \forall(\omega, \omega') \in \Omega \times \Omega.$

(iii) *There exists a measurable function Z on $(\Omega, \mathcal{F})$, a non-decreasing function f on $\mathbb{R}$, and a non-increasing function g on $\mathbb{R}$ such that $X = f(Z)$ and $Y = g(Z)$.*

(iv) *There exist two continuous functions u and v on $\mathbb{R}$ such that u is non-decreasing, v is non-increasing, $u(z) - v(z) = z, \ z \in \mathbb{R}$, and $X = u(X - Y)$, $Y = v(X - Y)$.*

In the first part of this subsection 1.4.2 we have exhibited an example of a pair (X, Y) of non-negative measurable functions for which the lower bound in the "generalized" Hardy-Littlewood's inequalities (theorem 1.4.1) is attained. It is easily observed that the measurable functions X and Y of that example are anti-comonotonic.
On the other hand, it is well-known that in the classical case where μ is a probability measure the lower bound in theorem 1.4.1 is attained by any pair of non-negative anti-comonotonic random variables. In the proposition which follows we establish that the converse statement also holds true : if the lower bound in theorem 1.4.1 is attained by any pair of non-negative anti-comonotonic measurable functions, then μ is a probability measure.

Proposition 1.4.2 *Let μ be a convex, continuous from below capacity on $(\Omega, \mathcal{F})$.
If $\mathbb{E}_\mu(XY) = \int_0^1 r_{X,\mu}(t) r_{Y,\mu}(1-t)dt$, for all pairs (X, Y) of non-negative anti-comonotonic measurable functions, then μ is a probability measure.*

In order to prove the proposition we will need the following observation :

Property 1.4.1 *Let μ be a concave (or a convex) capacity on $(\Omega, \mathcal{F})$. The following two assertions are equivalent :*

(i) *The capacity μ is additive (i.e. $\mu(A \cup B) = \mu(A) + \mu(B)$, for all $A, B \in \mathcal{F}$ such that $A \cap B = \varnothing$).*

(ii) *The capacity μ is equal to its dual capacity $\bar{\mu}$ (i.e. $\mu(A) = \bar{\mu}(A)$, $\forall A \in \mathcal{F}$).*

We note that the previous property 1.4.1 can be obtained by combining proposition 3 in Marinacci and Montrucchio (2004) with remark 4.90 in Föllmer and Schied (2004) concerning the core of a concave (or convex) capacity. We give a direct proof of the property for reader's convenience.

Proof of property 1.4.1 : We will prove the result for a *concave* μ (the proof in the case where μ is assumed to be convex is similar).

The implication $(i) \Rightarrow (ii)$ is due to the definitions ; its proof is straightforward.

To prove the converse implication, let us suppose that μ is equal to $\bar{\mu}$.

Being the dual of the concave capacity μ, the capacity $\bar{\mu}$ is convex. This observation, combined with the equality $\mu = \bar{\mu}$, leads to the conclusion that μ is convex. Thus, we obtain that μ is concave and convex ; the additivity of μ follows.

□

Proof of proposition 1.4.2 : We will show that $\mu(\cdot) = \bar{\mu}(\cdot)$. The conlusion will follow thanks to property 1.4.1, and to the continuity from below of μ.

Let $A \in \mathcal{F}$. Set $X := f(\mathbb{I}_A)$ and $Y := g(\mathbb{I}_A)$, where f and g are two functions on $\mathbb{R}$ such that f is non-negative non-increasing, g is non-negative non-decreasing, and fg is non-increasing. The measurable functions X and Y are anti-comonotonic. Therefore, by assumption,

$$\mathbb{E}_\mu(XY) = \int_0^1 r_{X,\mu}(t) r_{Y,\mu}(1-t)dt. \tag{1.4.7}$$

Let us compute explicitly $\mathbb{E}_\mu(XY)$ and $\int_0^1 r_{X,\mu}(t) r_{Y,\mu}(1-t)dt$. For that purpose, we note that

$$\begin{aligned} X &= f(1)\mathbb{I}_A + f(0)\mathbb{I}_{A^c} = f(1) + \big(f(0) - f(1)\big)\mathbb{I}_{A^c}, \\ Y &= g(1)\mathbb{I}_A + g(0)\mathbb{I}_{A^c} = g(0) + \big(g(1) - g(0)\big)\mathbb{I}_A, \text{ and} \\ XY &= f(1)g(1)\mathbb{I}_A + f(0)g(0)\mathbb{I}_{A^c} = f(1)g(1) + \big(f(0)g(0) - f(1)g(1)\big)\mathbb{I}_{A^c}. \end{aligned} \tag{1.4.8}$$

We note that the numbers $f(0)-f(1)$, $g(1)-g(0)$, and $f(0)g(0)-f(1)g(1)$ are non-negative due to the assumptions on f and g.

Thus, $\mathbb{E}_\mu(XY) = \mathbb{E}_\mu\Big(f(1)g(1) + \big(f(0)g(0) - f(1)g(1)\big)\mathbb{I}_{A^c}\Big)$. Thanks to the translation invariance and the positive homogeneity of the Choquet integral, we obtain

$$\begin{aligned} \mathbb{E}_\mu(XY) &= f(1)g(1) + \big(f(0)g(0) - f(1)g(1)\big)\mu(A^c) \\ &= f(1)g(1) + \big(f(0)g(0) - f(1)g(1)\big)(1 - \bar{\mu}(A)). \end{aligned} \tag{1.4.9}$$

In the computations of $r_{X,\mu}(t)$ and $r_{Y,\mu}(t)$ which follow, the equalities are to be taken in the almost everywhere sense.

We have

$$r_{X,\mu}(t) = f(1) + \big(f(0) - f(1)\big) r_{\mathbb{I}_{A^c},\mu}(t) = f(1) + \big(f(0) - f(1)\big)\mathbb{I}_{(1-\mu(A^c),1]}(t),$$

where the first equality is obtained by using the expression of X from equation (1.4.8), and by applying lemma 1.2.1 with the non-decreasing continuous function $x \mapsto f(1) + (f(0) - f(1))x$.
Similarly,

$$r_{Y,\mu}(t) = g(0) + (g(1) - g(0))r_{\mathbb{I}_A,\mu}(t) = g(0) + (g(1) - g(0))\mathbb{I}_{(1-\mu(A),1]}(t).$$

Using the expressions of $r_{X,\mu}(t)$ and $r_{Y,\mu}(t)$, as well as the fact that $\bar{\mu}(A) := 1 - \mu(A^c) \geq \mu(A)$ (which is due to the convexity of μ), we obtain

$$\int_0^1 r_{X,\mu}(t)r_{Y,\mu}(1-t)dt = f(0)g(0)(1-\bar{\mu}(A)) + f(1)(g(1) - g(0))\mu(A) + f(1)g(0)\bar{\mu}(A). \tag{1.4.10}$$

After replacing the expressions of $\mathbb{E}_\mu(XY)$ (from equation (1.4.9)) and of $\int_0^1 r_{X,\mu}(t)r_{Y,\mu}(1-t)dt$ (from equation (1.4.10)) in equation (1.4.7), and then simplifying and rearranging the terms, we obtain

$$f(1)(g(1) - g(0))(\bar{\mu}(A) - \mu(A)) = 0. \tag{1.4.11}$$

Let us choose f and g in such a manner that, along with the requirements already made, the conditions $f(1) > 0$ and $g(1) > g(0)$ are satisfied. For such f and g, the previous equality (1.4.11) implies $\bar{\mu}(A) = \mu(A)$.
The measurable set A in the previous reasoning being arbitrary, we obtain $\bar{\mu}(\cdot) = \mu(\cdot)$. This equality, combined with the convexity of μ, allows us to conclude that the capacity μ is additive (cf. property 1.4.1). The additivity and the continuity from below of the capacity μ imply that μ is a probability measure.

□

1.A Appendix

Proof of lemma 1.3.1 : The function f being non-decreasing, we define the following (upper) inverse $\check{f}$ of f by $\check{f}(y) := \sup\{z : f(z) \leq y\}$, $\forall y \in \mathbb{R}$. Note that according to remark 1.2.2 the function $\check{f}$ can be expressed in the following manner $\check{f}(y) := \inf\{z : f(z) > y\}$, $\forall y \in \mathbb{R}$. As the function f is non-decreasing and as the functions f and G_Z have no common discontinuities, we know from Yan (2009) that

$$G_{f(Z)}(x) = G_Z \circ \check{f}(x), \ \forall x \in \mathbb{R}. \tag{1.A.1}$$

Thanks to (1.A.1) and to remark 1.2.2, the upper quantile function $r^+_{f(Z)}$ of $f(Z)$ can be expressed as follows

$$r^+_{f(Z)}(t) = \sup\{x : G_Z \circ \check{f}(x) \leq t\} = \inf\{x : G_Z \circ \check{f}(x) > t\}. \tag{1.A.2}$$

For a fixed $t \in (0,1)$, let us first prove that $r^+_{f(Z)}(t) \geq f(r^+_Z(t))$ which, thanks to the previous considerations, amounts to showing that $\inf\{x : G_Z \circ \check{f}(x) > t\} \geq f(r^+_Z(t))$. The case where the set $\{x : G_Z \circ \check{f}(x) > t\}$ is empty being trivial, let $x \in \mathbb{R}$ be such that

$$G_Z \circ \check{f}(x) > t. \tag{1.A.3}$$

Now, the inequality (1.A.3) and the fact that $r^+_Z(t) = \inf\{y : G_Z(y) > t\}$ imply that $\check{f}(x) \geq r^+_Z(t)$. We consider two cases

- 1st case : If x is such that $\check{f}(x) > r^+_Z(t)$, then $f(r^+_Z(t)) \leq x$. This implication is due to the definition of $\check{f}(x)$.
- 2nd case : In the case where x is such that $\check{f}(x) = r^+_Z(t)$, the inequality (1.A.3) gives $G_Z(r^+_Z(t)) > t$.
  In the sub-case where $\check{f}(x)$ and $r^+_Z(t)$ belong to $\mathbb{R}$, we conclude from the latter inequality that $r^+_Z(t)$ is a point of discontinuity of G_Z which implies that f is continuous at $r^+_Z(t)$. Thus we obtain that $f(r^+_Z(t)) = f(\check{f}(x)) = x$.
  In the sub-case where $\check{f}(x) = r^+_Z(t) = +\infty$, we have, thanks to the definition of $\check{f}(x)$, that $\sup_{y\in\mathbb{R}} f(y) \leq x$. Therefore, $f(r^+_Z(t)) = f(+\infty) \leq x$.
  The measurable function Z being real-valued, the inequality (1.A.3) implies that $\check{f}(x) \neq -\infty$. Thus, only the two above-mentioned sub-cases are to be considered.

In both of the cases the inequality $x \geq f(r^+_Z(t))$ holds ; the desired inequality $r^+_{f(Z)}(t) \geq f(r^+_Z(t))$ follows.
Let us prove the converse inequality namely $r^+_{f(Z)}(t) \leq f(r^+_Z(t))$ which is equivalent to $\sup\{x : G_Z \circ \check{f}(x) \leq t\} \leq f(r^+_Z(t))$. Let x be such that $G_Z \circ \check{f}(x) \leq t$. This inequality implies that $\check{f}(x) \neq +\infty$ and that $\check{f}(x) \leq r^+_Z(t)$.

- If $\check{f}(x) \in \mathbb{R}$, then applying the non-decreasing function f at both sides of the latter inequality gives $f(\check{f}(x)) \leq f(r^+_Z(t))$. Now, the function f being right-continuous and the function $\check{f}$ being a generalized inverse of f we have $f(\check{f}(x)) = f(\check{f}(x)+) \geq x$. Thus we obtain $x \leq f(r^+_Z(t))$.
- If $\check{f}(x) = -\infty$, then $x \leq \inf_{y\in\mathbb{R}} f(y)$ (due to the definition of $\check{f}(x)$). Therefore, $x \leq f(r^+_Z(t))$ which concludes the proof.

□

Proof of proposition 1.3.1 : Let us prove the result concerning the upper quantile functions (equation (1.3.1)). The proof is based on lemma 1.3.1. The assertion concerning the lower quantile functions follows from lemma 1.3.2 by means of similar arguments.
According to proposition 1.2.1, there exist two non-decreasing continuous functions $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ and a real-valued measurable function Z such that $X = u(Z)$ and $Y = v(Z)$. Let $t \in (0,1)$. As the function $u + v$ is non-decreasing and continuous, we can

apply lemma 1.3.1 to obtain

$$r^+_{X+Y}(t) = r^+_{(u+v)(Z)}(t) = (u+v)r^+_Z(t) = u\left(r^+_Z(t)\right) + v\left(r^+_Z(t)\right).$$

It follows from lemma 1.3.1 (applied with $f = u$ and with $f = v$) that $u\left(r^+_Z(t)\right) = r^+_{u(Z)}(t)$ and $v\left(r^+_Z(t)\right) = r^+_{v(Z)}(t)$ which concludes the proof.

$\square$

CHAPITRE 2

Stochastic orderings with respect to a capacity and an application to a financial optimization problem

Abstract : By analogy with the classical case of a probability measure, we extend the notion of increasing convex (concave) stochastic dominance relation to the case of a normalized monotone (but not necessarily additive) set function also called a capacity. We give different characterizations of this relation establishing a link to the notions of distribution function and quantile function with respect to the given capacity. The Choquet integral is extensively used as a tool. In the second part of the chapter, we give an application to a financial optimization problem whose constraints are expressed by means of the increasing convex stochastic dominance relation with respect to a capacity. The problem is solved by using, among other tools, a result established in our previous work, namely a new version of the classical upper (resp. lower) Hardy-Littlewood's inequality generalized to the case of a continuous from below concave (resp. convex) capacity. The value function of the optimization problem is interpreted in terms of risk measures (or premium principles).

2.1 Introduction

Capacities and integration with respect to capacities were introduced by G. Choquet and were afterwards applied in different areas such as economics and finance among many others (cf. for instance Wang and Yan 2007 for an overview of applications). In economics

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and finance, capacities and Choquet integrals have been used, in particular, to build alternative theories to the "classical" setting of expected utility maximization of Von Neumann and Morgenstern. Indeed, the classical expected utility paradigm has been challenged by various empirical experiments and "paradoxes" (such as Allais's and Ellsberg's) thus leading to the development of new theories. One of the proposed new paradigms is the Choquet expected utility (abridged as CEU) where agent's preferences are represented by a capacity μ and a non-decreasing real-valued function u. The agent's "satisfaction" with a claim X is assessed by the Choquet integral of $u(X)$ with respect to the capacity μ. Choquet expected utility intervenes in situations where an objective probability measure is not given and where the agents are not able to derive a subjective probability over the set of different scenarios (cf., for instance, Schmeidler 1989, Chateauneuf 1994, Chateauneuf et al. 2000, for more details concerning the CEU-theory).
On the other hand, stochastic orders have also been extensively used in the decision theory. They represent partial order relations on the space of random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (more precisely, stochastic orders are partial order relations on the set of the corresponding distribution functions). Different kinds of orders have been studied and applied (see, for instance, Müller and Stoyan 2002, and Shaked and Shanthikumar 2006 for a general presentation) and links to the expected utility theory have been explored. Hereafter, we will call "classical" the results on stochastic orders in the case of random variables on a *probability* space. In the classical setting of random variables on a probability space, there are two approaches to risk orderings : economic ordering based on classes of utility functions, and statistical ordering which is based on tail distributions (cf. the explanations in Wang and Young 1998). In the "classical" case of a probability space, the two approaches lead to definitions which are equivalent. For the purpose of this chapter we will focus on the increasing convex ordering (or increasing convex stochastic dominance relation). The economic approach to the classical increasing convex stochastic dominance leads to the following definition - X is said to be dominated by Y in the increasing convex stochastic dominance relation (denoted $X \leq_{icx} Y$) if $\mathbb{E}(u(X)) \leq \mathbb{E}(u(Y))$ for all $u : \mathbb{R} \to \mathbb{R}$ non-decreasing and convex, provided the expectations (taken in the Lebesgue sense) exist in $\mathbb{R}$. The economic interpretation is then the following : the claim X is dominated by the claim Y in the increasing convex stochastic dominance if Y is preferred to X by all decision makers who prefer more wealth to less and who are risk-seeking. The statistical approach leads to the following *equivalent* definition : $X \leq_{icx} Y$ if $\int_x^{+\infty} \mathbb{P}(X > u)du \leq \int_x^{+\infty} \mathbb{P}(Y > u)du$, $\forall x \in \mathbb{R}$, provided the integrals exist in $\mathbb{R}$. Moreover, we have the following characterization which establishes a link between the *icx* ordering relation and stop-loss premia in reinsurance (cf. Dhaene et al. 2006) : $X \leq_{icx} Y$ if and only if $\mathbb{E}((X-b)_+) \leq \mathbb{E}((Y-b)_+)$, $\forall b \in \mathbb{R}$, provided the expectations exist in $\mathbb{R}$.
In the **first part** of this chapter, we generalize the notion of increasing convex stochastic

dominance to the case where the measurable space $(\Omega, \mathcal{F})$ is endowed with a given capacity μ which is not necessarily a probability measure, and we investigate generalizations of the previously mentioned results to this setting. Our definition of increasing convex stochastic dominance relation with respect to a capacity μ (denoted by $\leq_{icx,\mu}$) is motivated by the Choquet expected utility theory (it is a "CEU-based" stochastic dominance relation). Of course, in our case "ordinary" expectations (in the Lebesgue sense) have to be replaced by Choquet expectations. We obtain that characterizations analogous to the previously mentioned remain valid in our more general setting if we assume that the capacity μ has certain continuity properties (namely, continuity from below and continuity from above). Nevertheless, let us remark that in all proofs but one the assumption of continuity from below and from above is not needed.

In the **second part** of the chapter, we study a financial optimization problem inspired by the work of Dana (2005) (see also Dana and Meilijson 2003 and the references therein, Jouini and Kallal 2001, Dybvig 1987, as well as the work of Kusuoka 2001 for a related result). In Dana (2005), and Dana and Meilijson (2003), the following optimization problem is considered :

$$\begin{gathered}\text{Minimize } \mathbb{E}(ZC) \\ \text{under the constraints : } C \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) \text{ such that } X \leq_{icv} C\end{gathered} \tag{$\tilde{D}$}$$

where the abbreviation *icv* stands for the increasing concave ordering relation (with respect to the probability measure $\mathbb{P}$), the symbol $\mathbb{E}$ denotes the Lebesgue expectation (with respect to the probability measure $\mathbb{P}$), and where $Z \in L^1_+(\Omega, \mathcal{F}, \mathbb{P})$ and $X \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$ are given. Let us recall that the *icv-* stochastic dominance relation is defined similarly to the *icx-* stochastic dominance, the class of *non-decreasing convex* real-valued functions in the definition being replaced by the class of *non-decreasing concave* real-valued functions. The authors interpret the value function of the above problem $(\tilde{D})$ as being the minimal expenditure to get a contingent claim among those which dominate the contingent claim X in the increasing concave ordering. The value function of the problem is linked to the notion of risk measure as well.

By analogy with this problem we are interested in the following optimization problem, where we are given a (continuous from below concave) capacity μ and a non-negative numéraire Z :

$$\underset{C \in \mathcal{A}(X)}{\text{Maximize}} \; \mathbb{E}_{\mu}(ZC), \tag{D}$$

the symbol $\mathbb{E}_{\mu}$ denoting the Choquet integral with respect to μ and $\mathcal{A}(X)$ standing for the set of non-negative bounded measurable functions C which precede a given non-negative bounded measurable function X in the $\leq_{icx,\mu}$ −ordering (cf. section 2.4 for a precise formulation of the problem). The measurable function Z can be interpreted as a discount factor, and the objective functional $C \mapsto \mathbb{E}_{\mu}(ZC)$ can be interpreted as a given reference risk measure (or premium principle in insurance) in which the discount factor Z is taken into account. The importance of discounting in risk measurement has been highlighted by El Karoui and Ravanelli (2009). We recall that the usage of Choquet integrals as risk measures (or premium principles in insurance) is not new (cf. the review articles of Wang and Yan 2007, Dhaene et al. 2006, as well as the book of Föllmer and Schied 2004). We also recall that the well-known distortion risk measures (or distortion premium principles), studied by Wang et al. (1997) and Denneberg (1990), are particular cases of Choquet integrals (with respect to a capacity of the form $\psi \circ \mathbb{P}$ where ψ is a distortion function and $\mathbb{P}$ is a given probability measure). Choquet integrals have also been used as non-linear pricing functionals in finance (cf. Chateauneuf et al. 1996, as well as the review paper by Wang and Yan 2007 and the references therein). Some connections between non-linear pricing functionals and risk measures have been made in the work of Bion-Nadal (2009) and Klöppel and Schweizer (2007).

We give an interpretation of the value function of problem (D) in terms of a class of risk measures (or premium principles) which we call "generalized" distortion risk measures (or "generalized" distortion premium principles in insurance). A decision maker (an insurance company for instance) which is willing to take into account the initial reference risk measure $\mathbb{E}_\mu(Z\cdot)$, as well as other criteria of "riskiness" modelled by the $\leq_{icx,\mu} -$ relation, could use problem (D) as a way of devising a "new" risk measure (cf. section 2.4 for more details). In order to solve problem (D), we use the "generalized" version of Hardy-Littlewood's inequalities which we have obtained in the previous chapter (see also our note Grigorova 2013). We also provide a "dual" characterization of the value function of problem (D) as the smallest risk measure which is consistent with respect to the $\leq_{icx,\mu} -$ relation and which is greater than, or equal to, the initial reference risk measure.

The present chapter is based on our working paper Grigorova (2010).

The rest of the chapter is organized as follows. In section 2.2 we fix the terminology and the notation by recalling some well-known definitions about capacities and Choquet integrals ; in particular, the notions of comonotonic measurable functions and quantile function with respect to a capacity are recalled. In section 2.3 we define the notion of increasing convex (concave) stochastic dominance with respect to a capacity and explore different characterizations analogous to those existing in the classical case of a probability measure. In section 2.4 we formulate and solve our optimization problem (D) ; in subsection 2.4.1 we provide an interpretation of the value function in terms of risk measures in finance (or premium principles in insurance) ; in subsection 2.4.2 we give a "dual" characterization of the value function of problem (D). Finally, in section 2.5 we briefly present a part of our subsequent research concerning some related questions. The Appendix contains two parts : in Appendix 2.A some complements on Choquet integration are given : they are used in the proof of one of the characterizations of the $\leq_{icx,\mu}$ –relation ; Appendix 2.B is devoted to the proofs of a lemma and a proposition from section 2.3 which are similar to the proofs of the corresponding "classical" results.

2.2 Notation, definitions and some basic properties

The definitions and results recalled in this section can be found in the book by Denneberg (1994), and/or in that by Föllmer and Schied (2004) (cf. section 4.7).
Let $(\Omega, \mathcal{F})$ be a measurable space. We denote by χ the space of measurable, real-valued and bounded functions on $(\Omega, \mathcal{F})$.

Definition 2.2.1 *A set function $\mu : \mathcal{F} \longrightarrow [0,1]$ is called a* capacity *if it satisfies $\mu(\varnothing) = 0$ (groundedness), $\mu(\Omega) = 1$ (normalization) and the following monotonicity property : $A, B \in \mathcal{F}, A \subset B \Rightarrow \mu(A) \leq \mu(B)$.*

Definition 2.2.2 *A capacity μ is called* concave *(or submodular, or 2-alternating) if*
$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$, *for all $A, B \in \mathcal{F}$.*
A capacity μ is called convex *(or supermodular) if it satisfies the previous property where the inequality is reversed.*
A capacity μ is called continuous from below *if*

$$(A_n) \subset \mathcal{F} \text{ such that } A_n \subset A_{n+1}, \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\cup_{n=1}^{\infty} A_n).$$

A capacity μ is called continuous from above *if*

$$(A_n) \subset \mathcal{F} \text{ such that } A_n \supset A_{n+1}, \forall n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n).$$

The dual *capacity $\bar{\mu}$ of a given capacity μ is defined by*

$$\bar{\mu}(A) := 1 - \mu(A^c), \text{ for all } A \in \mathcal{F}.$$

We recall the notions of (non-decreasing) distribution function and of a quantile function with respect to a capacity μ (cf. Föllmer and Schied 2004).

Definition 2.2.3 *Let X be a measurable function on $(\Omega, \mathcal{F})$. We define the distribution function G_X of X with respect to μ by $G_X(x) := 1 - \mu(X > x)$, for all $x \in \bar{\mathbb{R}}$.*
Any generalized inverse *function $r_X : (0,1) \longrightarrow \bar{\mathbb{R}}$ of the* non-decreasing *function G_X is called a* quantile *function of X with respect to μ.*

For notational convenience, we omit the dependence on μ in the notation G_X and r_X when the omission is not misleading.

Remark 2.2.1 Let μ be a capacity and let X be a measurable real-valued function such that

$$\lim_{x \to -\infty} G_X(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} G_X(x) = 1. \tag{2.2.1}$$

We denote by $G_X(x-)$ and $G_X(x+)$ the left-hand and right-hand limits of G_X at x. A function r_X is a quantile function of X (with respect to μ) if and only if

$$G_X(r_X(t)-) \leq t \leq G_X(r_X(t)+), \quad \forall t \in (0,1).$$

In this case r_X is real-valued. Note that the condition (2.2.1) is satisfied if $X \in \chi$ and μ is arbitrary. The condition (2.2.1) is satisfied for an arbitrary X if μ is continuous from below and from above.

We recall the notion of comonotonic functions (cf. Föllmer and Schied 2004).

Definition 2.2.4 *Two real-valued measurable functions X and Y on $(\Omega, \mathcal{F})$ are called* comonotonic *if*

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0, \ \forall (\omega, \omega') \in \Omega \times \Omega.$$

For a measurable function X on $(\Omega, \mathcal{F})$, the *Choquet integral* of X with respect to a capacity μ is defined as follows :

$$\mathbb{E}_\mu(X) := \int_0^{+\infty} \mu(X > x)dx + \int_{-\infty}^0 (\mu(X > x) - 1)dx.$$

Note that the Choquet integral in the preceding definition may not exist (namely, if one of the two (Riemann) integrals on the right-hand side is equal to $+\infty$ and the other to $-\infty$), may be in $\mathbb{R}$ or may be equal to $+\infty$ or $-\infty$. The Choquet integral always exists if the function X is bounded from below or from above. The Choquet integral exists and is finite if X is in χ.

For other well-known results about Choquet integrals, quantile functions with respect to a capacity and comonotonic functions which will be used in the sequel the reader is referred to section 1.2 of the previous chapter 1.

We end this section by two examples of a capacity. The first example is well-known in the decision theory (think for instance of the rank-dependent expected utility theory - Quiggin 1982, or of Yaari's distorted utility theory in Yaari 1997) ; the second is a slight generalization of the first and can be found in Denneberg (1994).

Example 2.2.1 1. Let P be a probability measure on $(\Omega, \mathcal{F})$ and let $\psi : [0,1] \to [0,1]$ be a non-decreasing function on $[0,1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$. Then the set function $\psi \circ P$ defined by $\psi \circ P(A) := \psi(P(A)), \forall A \in \mathcal{F}$ is a capacity in the sense of definition 2.2.1. The function ψ is called a distortion function and the capacity $\psi \circ P$ is called a *distorted probability.* If the distortion function ψ is concave, the capacity $\psi \circ P$ is a concave capacity in the sense of definition 1.2.2.

2. Let μ be a capacity on $(\Omega, \mathcal{F})$ and let ψ be a distortion function. Then the set function $\psi \circ \mu$ is a capacity which, by analogy with the previous example, will be called a *distorted capacity.* Moreover, we have the following property : if μ is a concave capacity and ψ is concave, then $\psi \circ \mu$ is concave. The proof uses the same arguments as the proof of proposition 4.69 in Föllmer and Schied (2004) and is left to the reader (see also exercice 2.10 in Denneberg 1994).

2.3 Stochastic orderings with respect to a capacity

The aim of this section is to "extend" the concept of stochastic orderings from the "classical" case where the underlying measurable space is endowed with a probability measure to the more general case where the underlying measurable space is endowed with a capacity (which is not necessarily additive) ; for the purposes of this chapter, the stress is placed on the generalizations to the case of a capacity of the results on the increasing convex and the increasing concave stochastic dominance relations. As is usually done in the classical case, we emphasize the links between an economic approach to stochastic orderings based on numerical representations of the economic agents' preferences and a statistical approach based on a pointwise comparison of the distribution functions or of some other performance functions constructed from the distribution functions. Our definitions are analogous to the "classical" ones.

2.3.1 The increasing convex stochastic dominance with respect to a capacity μ

Analogously to the "classical" definition of increasing convex stochastic dominance (with respect to a probability measure), we define the notion of increasing convex stochastic dominance relation (or, equivalently, increasing convex ordering) with respect to a capacity μ as follows :

Definition 2.3.1 *Let X and Y be two real-valued measurable functions on $(\Omega, \mathcal{F})$ and let μ be a capacity on $(\Omega, \mathcal{F})$. We say that X is smaller than Y in the increasing convex ordering (with respect to the capacity μ) denoted by $X \leq_{icx} Y$ if*

$$\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$$

for all functions $u : \mathbb{R} \to \mathbb{R}$ which are non-decreasing and convex, provided the Choquet integrals exist in $\mathbb{R}$.

This definition coincides with the usual definition of the increasing convex order when the capacity μ is a probability measure on $(\Omega, \mathcal{F})$ (cf. Shaked and Shanthikumar 2006 for details in the classical case).

Remark 2.3.1 The economic interpretation of the *icx* ordering with respect to a capacity μ is the following : $X \leq_{icx} Y$ if all the CEU-maximizers whose preferences are described by the (common) capacity μ and a non-decreasing convex utility function (and who associate a real number to their satisfaction with X and Y) prefer the claim Y to the claim X. As explained in Kaas et al. (2001), the "classical" stochastic orderings allow to compare risks (or financial positions, or *gains*) according to the expected utility (EU) paradigm. The stochastic orderings with respect to a capacity studied here allow to compare financial positions according to the Choquet expected utility (CEU) theory. The $\leq_{icx,\mu}$ relation and the $\leq_{icv,\mu}$ relation (defined below) derive from the CEU theory as the corresponding "classical" stochastic orderings derive from the EU theory. Similarly to the "classical" case, we exclude from the comparison those CEU-maximizers who cannot evaluate either X or Y, and those who are infinitely satisfied or infinitely dissatisfied with X or Y.
Let us mention that an economic setting where all the agents are CEU-maximizers characterized by a common capacity μ and a non-decreasing convex (resp. concave) utility function has already been considered in the literature in the study of Pareto-optima (cf. Chateauneuf et al. 2000).
If the measurable functions X and Y are interpreted as *losses* (which will be the case in section 2.4), the increasing convex stochastic dominance with respect to a capacity μ can be interpreted as follows : $X \leq_{icx,\mu} Y$ if all the CEU-*minimizers* whose preferences are described by the (common) capacity μ and a non-decreasing convex *"pain"* function (see Denuit et al. 1999 for the terminology), and who associate a real number to their *dissatisfaction* with X and Y, prefer *losing* X to *losing* Y.

For the sake of completeness, we define the notion of an increasing concave stochastic dominance (or equivalently an increasing concave ordering) with respect to a capacity μ.

Definition 2.3.2 *Let X and Y be two real-valued measurable functions on $(\Omega, \mathcal{F})$ and let μ be a capacity on $(\Omega, \mathcal{F})$. We say that X is smaller than Y in the increasing concave ordering (with respect to the capacity μ) denoted by $X \leq_{icv} Y$ if*

$$\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$$

for all functions $u : \mathbb{R} \to \mathbb{R}$ which are non-decreasing and concave, provided the Choquet integrals exist in $\mathbb{R}$.

As in the previous section, the dependence on the capacity μ in the notation for the stochastic dominance relations $\leq_{icx}$ and $\leq_{icv}$ is intentionally omitted. Nevertheless, we shall note $\leq_{icx,\mu}$ and $\leq_{icv,\mu}$ when an explicit mention of the capacity to which we refer is needed.
The ordering relations of the previous two definitions are linked to each other in the following manner :

Proposition 2.3.1 *Let X and Y be two real-valued measurable functions. The following equivalence holds true :*

$$X \leq_{icx,\mu} Y \Leftrightarrow -Y \leq_{icv,\bar{\mu}} -X$$

where $\bar{\mu}$ denotes the dual capacity of the capacity μ.

Proof: The proof is based on the fact that a function $x \mapsto u(x)$ is non-decreasing and convex in x if and only if the function $x \mapsto -u(-x)$ is non-decreasing and concave in x, and on the property of asymmetry of the Choquet integral ; the details are straightforward. □

We note that in the classical case where the capacity μ is a probability measure, the dual $\bar{\mu}$ is equal to μ ; so, in that case, the previous proposition 2.3.1 is reduced to a well-known result from the stochastic order literature (cf. theorem 4.A.1. of Müller and Stoyan 2002).
The aim of the following propositions is to obtain characterizations of the stochastic dominance relations $\leq_{icx}$ and $\leq_{icv}$. Due to proposition 2.3.1, we need to consider the case of $\leq_{icx}$ only.

Proposition 2.3.2 *Let μ be a capacity. We have the following statements :*

(i) *If $X \leq_{\mathrm{icx},\mu} Y$, then $\mathbb{E}_\mu((X-b)_+) \leq \mathbb{E}_\mu((Y-b)_+),\ \forall b \in \mathbb{R}$, provided the Choquet integrals exist in $\mathbb{R}$.*

(ii) *If the capacity μ has the additional properties of continuity from below and continuity from above, then the converse implication holds true, namely :
if $\mathbb{E}_\mu((X-b)_+) \leq \mathbb{E}_\mu((Y-b)_+),\ \forall b \in \mathbb{R}$, provided the Choquet integrals exist in $\mathbb{R}$, then $X \leq_{\mathrm{icx},\mu} Y$.*

Proof: The proof is an adaptation of the proof of theorem 1.5.7. in Müller and Stoyan (2002) to our case.
The proof of assertion (i) is trivial, the function $x \mapsto (x-b)_+$ being non-decreasing and convex for all $b \in \mathbb{R}$.

Let us now prove the assertion (ii). Let u be a non-decreasing and convex function *such that* $\mathbb{E}_\mu(u(X))$ *exists in* $\mathbb{R}$ *and* $\mathbb{E}_\mu(u(Y))$ *exists in* $\mathbb{R}$. We consider three cases :

1. The case where $\lim_{x\to-\infty} u(x) = 0$. It is well-known that u can be approximated from below by a sequence (u_n) of functions of the following form (cf., for instance, Müller and Stoyan 2002) :

$$u_n(x) = \sum_{i=1}^{n2^n} a_{in}(x - b_{in})_+,$$

where $a_{in} \geq 0$ and $b_{in} \in \mathbb{R}\cup\{+\infty\}$. Let us remark that all the functions in the family $(a_{in}(X-b_{in})_+)_{i\in\{1,\cdots,n2^n\}}$ are pairwise comonotonic (thanks to proposition 1.2.1) ; so, for all $i \in \{2, \cdots, n2^n\}$, $a_{in}(X-b_{in})_+$ is comonotonic with $\sum_{j=1}^{i-1} a_{jn}(X-b_{jn})_+$. Using the properties of comonotonic additivity and positive homogeneity of the Choquet integral, and a reasoning by induction, we obtain $\mathbb{E}_\mu(u_n(X)) = \sum_{i=1}^{n} a_{in}\mathbb{E}_\mu[(X - b_{in})_+]$. The same holds when X is replaced by Y. Thus,

$$\mathbb{E}_\mu(u_n(X)) = \sum_{i=1}^{n} a_{in}\mathbb{E}_\mu[(X - b_{in})_+] \leq \sum_{i=1}^{n} a_{in}\mathbb{E}_\mu[(Y - b_{in})_+] = \mathbb{E}_\mu(u_n(Y)).$$

The capacity μ being continuous from below, we apply the monotone convergence theorem (recalled in theorem 1.2.1) in order to pass to the limit in the previous inequality ; thus, we obtain $\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$.

2. The case where $\lim_{x\to-\infty} u(x) = a \in \mathbb{R}$ can be reduced to the previous one by considering the function $x \mapsto u(x) - a$. Thanks to point 1 we obtain $\mathbb{E}_\mu(u(X) - a) \leq \mathbb{E}_\mu(u(Y) - a)$. We conclude thanks to the translation invariance of the Choquet integral.

3. The case where $\lim_{x\to-\infty} u(x) = -\infty$. For $n \in \mathbb{N}$, we define the function u_n by $u_n(x) := \max(u(x), -n)$. The sequence (u_n) decreases to u (i.e. $u_n \downarrow u$). Moreover, the function u_n fulfils the conditions of the second case (we note that u_n is non-decreasing, convex and bounded from below). So,

$$\mathbb{E}_\mu(u_n(X)) \leq \mathbb{E}_\mu(u_n(Y)), \quad \text{for all } n \in \mathbb{N}. \tag{2.3.1}$$

We can pass to the limit in equation (2.3.1) thanks to proposition 2.A.1 of the appendix. More precisely, by applying the second statement of proposition 2.A.1 with $Z := u(X)$, $Z_n := u_n(X)$ and the capacity μ (which is continuous from above by assumption), we obtain $\mathbb{E}_\mu(u_n(X)) \downarrow \mathbb{E}_\mu(u(X))$. We note that the assumption of "integrability with respect to μ" of $Z = u(X)$ and $Z_0 = u_0(X)$ of proposition 2.A.1 is satisfied : indeed, the integral $\mathbb{E}_\mu(u(X))$ exists and is finite due to the assumption on u ; the integral $\mathbb{E}_\mu(Z_0)$ exists and is finite due to the definition of u_0 and to the "integrability with respect to μ" of $u(X)$.

By the same arguments we obtain $\mathbb{E}_\mu(u_n(Y)) \downarrow \mathbb{E}_\mu(u(Y))$. These two observations combined with equation (2.3.1) allow us to conclude that $\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$.

□

Remark 2.3.2 We note that in the particular case where the measurable functions X and Y are bounded, the third step of the previous proof could be simplified. Indeed, if X is bounded, then $(u_n(X))$ is a bounded sequence (in fact, it can be easily seen that $\max\left(u(\sup X), 0\right) \geq u_n(X) \geq u(\inf X)$, for all n, where $\inf X$ and $\sup X$ denote the lower and upper bound of X, respectively). Thanks to the monotone convergence theorem (theorem 1.2.1) and to the translation invariance of the Choquet integral, we obtain $\lim_{n\to\infty} \mathbb{E}_\mu(u_n(X)) = \mathbb{E}_\mu(u(X))$. The same observation holds when X is replaced by Y. These observations and equation (2.3.1) give the desired conclusion, namely $\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$.

In the classical case where μ is a probability measure the previous proposition 2.3.2 is reduced to a well-known characterization of the increasing convex order : it allows to link the increasing convex order to the notion of stop-loss premium in reinsurance. Accordingly, in the classical case the increasing convex order is also called stop-loss order.

Let us now establish a link between the increasing convex stochastic dominance with respect to a capacity μ and the notion of distribution function with respect to the capacity μ.

Proposition 2.3.3 *Let μ be a capacity and let X and Y be two measurable functions. The following two statements are equivalent :*

(i) $\mathbb{E}_\mu((X-b)_+) \leq \mathbb{E}_\mu((Y-b)_+)$, $\forall b \in \mathbb{R}$,
provided the Choquet integrals exist in $\mathbb{R}$.

(ii) $\int_x^{+\infty} \mu(X > u)du \leq \int_x^{+\infty} \mu(Y > u)du$, $\forall x \in \mathbb{R}$,
provided the integrals exist in $\mathbb{R}$.

Proof: Using the definition of the Choquet integral and a change of variables, we have for all $b \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}_\mu((X-b)_+) &= \int_0^{+\infty} \mu((X-b)_+ > u)du = \int_0^{+\infty} \mu(X > u+b)du \\ &= \int_b^{+\infty} \mu(X > u)du,\end{aligned}$$

which proves the desired result. □

We are ready to link the previous results to the notion of a quantile function with respect to μ. We refer the reader to Shaked and Shanthikumar (2006) for a proof of the following result in the classical case of a probability measure, and to Ogryczak and Ruszczynski (2001) for a different proof of the same result based on convex duality ; the reader is also referred to lemma A.22 in Föllmer and Schied (2004). Our proof is inspired by the last two references.

Proposition 2.3.4 *Let μ be a capacity and let X and Y be two real-valued measurable functions such that $\int_0^1 |r_X(t)|dt < +\infty$ and $\int_0^1 |r_Y(t)|dt < +\infty$ where r_X and r_Y denote (the) quantile functions of X and Y with respect to μ. The following two statements are equivalent :*

(i) $G_X^{(2)}(x) := \int_x^{+\infty} \mu(X > u)du \leq \int_x^{+\infty} \mu(Y > u)du =: G_Y^{(2)}(x), \forall x \in \mathbb{R}$.

(ii) $\int_y^1 r_X(t)dt \leq \int_y^1 r_Y(t)dt, \forall y \in [0, 1]$.

In order to prove the above proposition we need the following lemma, which corresponds to lemma A.22 in Föllmer and Schied (2004) in the classical case.

Lemma 2.3.1 *Let μ be a capacity on $(\Omega, \mathcal{F})$ and let X be a measurable function on $(\Omega, \mathcal{F})$ such that the quantile function r_X of X with respect to μ is integrable (with respect to the Lebesgue measure on $[0, 1]$). Define the function $G_X^{(2)}$ by*

$$G_X^{(2)}(x) := \int_x^{+\infty} \mu(X > u)du = \int_x^{+\infty} (1 - G_X(u))du, \ x \in \mathbb{R}.$$

The conjugate function $r_X^{(2)}$ of the function $G_X^{(2)}$ is given by

$$r_X^{(2)}(y) := \sup_{x \in \mathbb{R}} (xy - G_X^{(2)}(x)) = \begin{cases} -\int_{y+1}^1 r_X(t)dt, & \textit{if } y \in [-1, 0] \\ +\infty, & \textit{otherwise.} \end{cases}$$

Proof of the lemma : The arguments of the proof being almost the same as those of Föllmer and Schied (2004), the proof is placed in the Appendix 2.B.

We are ready to prove proposition 2.3.4.

Proof of proposition 2.3.4 :

The proof is based on lemma 2.3.1.

Suppose that (i) holds true, i.e. $G_X^{(2)}(x) \leq G_Y^{(2)}(x)$, for all $x \in \mathbb{R}$. Then, for all $y \in \mathbb{R}$,

$$r_X^{(2)}(y) := \sup_{x \in \mathbb{R}} (xy - G_X^{(2)}(x)) \geq \sup_{x \in \mathbb{R}} (xy - G_Y^{(2)}(x)) =: r_Y^{(2)}(y),$$

which implies, in particular, that $-\int_{y+1}^1 r_X(t)dt \geq -\int_{y+1}^1 r_Y(t)dt$, for all $y \in [-1, 0]$, or equivalently,

$$\int_y^1 r_X(t)dt \leq \int_y^1 r_Y(t)dt, \text{ for all } y \in [0, 1].$$

The converse implication can be obtained by means of a similar argument after observing that the function $G_X^{(2)}$ is the conjugate function of $r_X^{(2)}$. This observation follows from the fact that the function $G_X^{(2)}$ is convex, proper and lower-semicontinuous (cf. theorem 24.2 in Rockafellar 1972) and from the biduality theorem (cf. theorem 12.2 in Rockafellar 1972). □

We conclude this section by establishing another useful characterization of the relation $\leq_{icx}$ which will be needed in the sequel. Its analogue in the classical case of a probability measure can be found in Dana (2005) (see also thm. 5.2.1 in Dhaene et al. 2006 for a related result). Our proof follows the proof of the former.

Proposition 2.3.5 *Let $X \in \chi$ and $Y \in \chi$ be given. The following statements are equivalent :*

(i) $\int_y^1 r_X(t)dt \geq \int_y^1 r_Y(t)dt, \forall y \in [0,1]$

(ii) $\int_0^1 g(t)r_X(t)dt \geq \int_0^1 g(t)r_Y(t)dt,\ \forall g : [0,1] \to \mathbb{R}_+$, *integrable, non-decreasing.*

Proof : Being similar to the proof of Dana (2005), the proof is given in the Appendix 2.B.

Remark 2.3.3 An economic interpretation of the $\leq_{icx,\mu}$ −relation in terms of "uniform" preferences is given in remark 2.3.1 ; the interpretation is based on the initial definition of the $\leq_{icx,\mu}$-relation (definition 2.3.1).
An interpretation of the $\leq_{icx,\mu}$ −relation in terms of ambiguity is suggested by the equivalence established in proposition 2.3.3. Indeed, let us first consider the inequality $\mu(X > t) \leq \mu(Y > t)$ where $t \in \mathbb{R}$ is fixed. Bearing in mind that the capacity μ models the agent's perception of "uncertain" (or "ambiguous") events, the reader may interpret the previous inequality as having the following meaning : the event $\{Y > t\}$ is perceived by the agent as being less uncertain than, or equally uncertain to, the event $\{X > t\}$. Then, part (ii) in proposition 2.3.3 may be loosely read as follows : the agent "feels less or equally uncertain about the financial position Y's taking great values on average than the financial position X's".

2.4 Application to a financial optimization problem

This section is devoted to the following optimization problem :

$$\begin{aligned} &\text{Maximize } \mathbb{E}_\mu(ZC) \\ &\text{under the constraints } C \in \chi_+ \text{ s.t. } C \leq_{icx,\mu} X \end{aligned} \tag{D}$$

where χ_+ denotes the set of non-negative bounded measurable functions, μ is a given capacity, Z is a given function in χ_+, and X is a given function in χ_+.

The study of this problem has been inspired by the work of Dana (2005) in the classical case of a probability measure ; see also Dana and Meilijson (2003), Jouini and Kallal (2001) and Dybvig (1987).
The following economic interpretation of problem (D) may be given. We place ourselves in a world where the agents are facing "ambiguous events" and we assume that all the agents perceive ambiguity in the same manner, i.e. through the same capacity μ. The objective functional $C \mapsto \mathbb{E}_\mu(ZC)$ can be interpreted as a (non-decreasing non-additive) premium principle, and the non-negative measurable function Z can be seen as a discount factor or, more generally, a "change of numéraire". We recall that in the insurance literature premium principles are functionals on χ_+ (or on χ) taking values in $\mathbb{R}$; these functionals are usually non-decreasing. This non-decreasingness requirement is due to the interpretation of the elements of χ_+ as payments which an insurance company has to make (or losses it has to face). In the case where the capacity μ is concave (which will be the case later on), the objective functional $C \mapsto \mathbb{E}_\mu(ZC)$ is convex. We note that a functional of this form (in the case $Z \equiv 1$) is used in Chateauneuf et al. (1996) in order to model the selling price of a claim (its buying price being modelled by $E_{\bar{\mu}}(\cdot)$). We note as well that, up to a minus sign, the objective functional is an example of a "cash-subadditive risk measure" in the terminology of El Karoui and Ravanelli (2009).
We consider a decision maker (an insurance company for instance) which uses the premium principle $\mathbb{E}_\mu(Z\cdot)$ as a reference premium principle, but which is now willing to devise a "new" premium principle which takes into account the preferences of a class of agents (aggregated by means of the $\leq_{icx,\mu}$ − relation). Thus, for a given loss $X \in \chi_+$, problem (D) consists in maximizing the initial premium principle $\mathbb{E}_\mu(Z\cdot)$ over the set of (non-negative) losses C which are dominated by X in the $\leq_{icx,\mu}$ − sense, i.e. which are "uniformly" preferred to X in the sense of the $\leq_{icx,\mu}$ − relation (cf. Remark 2.3.1).
Adopting the terminology introduced by Jouini and Kallal (2001), we may call the value function $e(X, Z)$ of problem (D) (when Z is fixed) the "utility premium" of X (or "pain premium" of X) in the context of ambiguity. It will be shown in subsection 2.4.2 that, for a fixed Z, the "utility premium" in the context of ambiguity $e(\cdot, Z)$ is the smallest functional on χ_+ among those which are consistent with respect to the $\leq_{icx,\mu}$ −relation and which are greater than or equal to the initial premium principle $\mathbb{E}_\mu(Z\cdot)$.
The following theorem holds true. In the proof we use the "generalized" Hardy-Littlewood's inequalities (cf. theorem 1.4.1 of the previous chapter, or Grigorova 2013).

Theorem 2.4.1 *Let μ be a concave and continuous from below capacity. For every $X \in \chi_+$ and for every $Z \in \chi_+$ such that the distribution function G_Z of Z with respect to μ is* *continuous, the problem*

$$\begin{aligned} &\textit{Maximize } \mathbb{E}_\mu(ZC) \\ &\textit{under the constraints } C \in \chi_+ \textit{ s.t. } C \leq_{\mathrm{icx},\mu} X \end{aligned} \tag{D}$$

has a solution and its value function $e(X,Z)$ *is given by :* $e(X,Z) = \int_0^1 r_Z(t) r_X(t) dt$.

Proof: We have

$$e(X,Z) = \sup_{C \in \chi_+, C \leq_{icx,\mu} X} \mathbb{E}_\mu(ZC) \leq \sup_{C \in \chi_+, C \leq_{icx,\mu} X} \int_0^1 r_Z(t) r_C(t) dt \leq \int_0^1 r_Z(t) r_X(t) dt$$

where the first inequality is due to the upper bound in Hardy-Littlewood's inequalities (theorem 1.4.1), and the second inequality is a consequence of proposition 2.3.5 (applied with $g = r_Z$).

Thus we obtain that $e(X,Z) \leq \int_0^1 r_Z(t) r_X(t) dt$. To conclude we need to find $C \in \chi_+$ such that $C \leq_{icx,\mu} X$ and such that $\mathbb{E}_\mu(ZC) = \int_0^1 r_Z(t) r_X(t) dt$.

Set $f(x) := r_X(G_Z(x))$, then $C := f(Z)$ is as wanted. Indeed, $C \geq 0$. Moreover,

$$\mathbb{E}_\mu(ZC) = \mathbb{E}_\mu(Zf(Z)) = \mathbb{E}_\mu(h(Z)) = \int_0^1 r_{h(Z)}(t) dt,$$

where we have used proposition 1.2.4 to obtain the last equality, and where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $h(z) := zf(z), \forall z \geq 0$. The function h being non-decreasing and the function G_Z being continuous by assumption, we can apply lemma 1.2.1 to obtain

$$\begin{aligned} \mathbb{E}_\mu(ZC) &= \int_0^1 h(r_Z(t)) dt = \int_0^1 r_Z(t) f(r_Z(t)) dt \\ &= \int_0^1 r_Z(t) r_X(G_Z(r_Z(t))) dt = \int_0^1 r_Z(t) r_X(t) dt, \end{aligned} \tag{2.4.1}$$

where we have used the continuity of G_Z in the last step.

We are left with establishing that $f(Z) \leq_{icx,\mu} X$. We will check this property by using the definition of $\leq_{icx,\mu}$. Let u be a non-decreasing, convex function. We have

$$\mathbb{E}_\mu(u(f(Z))) = \int_0^1 r_{u(f(Z))}(t) dt = \int_0^1 u(f(r_Z(t))) dt \tag{2.4.2}$$

where the second equality follows from lemma 1.2.1 (the function $u \circ f$ being non-decreasing and the function G_Z being continuous by assumption). This gives

$$\begin{aligned} \mathbb{E}_\mu(u(f(Z))) &= \int_0^1 u(r_X(G_Z(r_Z(t)))) dt = \int_0^1 u(r_X(t)) dt \\ &= \int_0^1 r_{u(X)}(t) dt = \mathbb{E}_\mu(u(X)) \end{aligned} \tag{2.4.3}$$

where the last but one equality is obtained thanks to lemma 1.2.1 after observing that u is a continuous function as a real-valued convex function on $\mathbb{R}$.

This concludes the proof. □

Remark 2.4.1 The previous proof can be extended to the case where the assumption of boundedness from above of Z is replaced by the weaker assumption $\int_0^1 |r_Z(t)|dt < +\infty$. This is due mainly to proposition 2.3.5 where only the non-negativity and the integrability of r_Z are required. We have nevertheless chosen to present the previous result in the case where all the functions are in χ_+.

In the classical case where μ is a probability measure the result of theorem 2.4.1 still holds even when the continuity assumption on the distribution function G_Z of Z is relaxed. More precisely, we have the following result :

Proposition 2.4.1 *Let μ be a probability measure on $(\Omega, \mathcal{F})$. For every function $X \in \chi_+$ and for every function $Z \in \chi_+$, the problem*

$$\textit{Maximize } \mathbb{E}(ZC)$$

$$\textit{under the constraints } C \in \chi_+ \textit{ s.t. } C \leq_{\mathrm{icx}} X$$

has a solution and its value function is given by $\int_0^1 r_Z(t)r_X(t)dt$.
The symbol $\mathbb{E}$ denotes the (classical) expectation with respect to μ and $\leq_{\mathrm{icx}}$ denotes the (classical) increasing convex stochastic dominance relation with respect to μ.

Proof: We sketch the proof by following the proof of theorem 2.4.1 and by stressing only on the changes to be made in the proof of theorem 2.4.1. Note that applying lemma 1.2.1 is still possible whenever needed in this case (even without the continuity assumption on G_Z) thanks to remark 1.2.4. Nevertheless, the continuity of G_Z being used to obtain the last equality in equation (2.4.1), the function f in the proof of theorem 2.4.1 is now replaced by the function $\tilde{f}$ defined by $\tilde{f}(x) := r_X(G_Z(x))$ if x is a continuity point of G_Z and by $\tilde{f}(x) := \frac{1}{G_Z(x)-G_Z(x-)}\int_{G_Z(x-)}^{G_Z(x)} r_X(t)dt$ if x is not a continuity point of G_Z. The function $\tilde{f}$ is non-decreasing and satisfies the property $\tilde{f}(r_Z) = \mathbb{E}_\lambda(r_X|r_Z)$ where the symbol $\mathbb{E}_\lambda(.|.)$ denotes the conditional expectation with respect to the Lebesgue measure λ.
We set $\tilde{h}(x) := x\tilde{f}(x)$ and we replace equation (2.4.1) by the following

$$\mathbb{E}(ZC) = \int_0^1 \tilde{h}(r_Z(t))dt = \int_0^1 r_Z(t)\tilde{f}(r_Z(t))dt = \int_0^1 r_Z(t)r_X(t)dt.$$

where lemma 1.2.1 and remark 1.2.4 are used to obtain the first equality and the characterization of the conditional expectation is used to obtain the last.
Equation (2.4.2) remains unchanged, the function f being replaced by the function $\tilde{f}$; we have again applied lemma 1.2.1 and remark 1.2.4 to obtain it.
Equation (2.4.3) has to be replaced by

$$\mathbb{E}(u(f(Z))) = \int_0^1 u(\tilde{f}(r_Z(t)))dt \leq \int_0^1 u(r_X(t))dt = \int_0^1 r_{u(X)}(t)dt = \mathbb{E}(u(X)),$$

where we have applied Jensen's inequality. $\square$

Remark 2.4.2 The previous proposition 2.4.1 is analogous to theorem 2.1 in Dana (2005), where the assumption of non-atomicity on the underlying probability space is made. We note that in the case where the underlying probability space $(\Omega, \mathcal{F}, \mu)$ is atomless the use of lemma 1.2.1 (and remark 1.2.4) in the proof of proposition 2.4.1 can be replaced by the use of the following two usual arguments : the law invariance of the functional $\mathbb{E}(l(\cdot)) : \chi_+ \to \mathbb{R}_+$ where $l : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable function, and the fact that the law of Z is the same as the law of $r_Z(U)$ where U denotes a uniform random variable on $(0,1)$. Then, the proof of proposition 2.4.1 becomes almost the same as the proof of theorem 2.1 in Dana (2005) (see also Dana and Meilijson 2003, and Föllmer and Schied 2004). We emphasize that the use of lemma 1.2.1 allows to prove proposition 2.4.1 without making an assumption of non-atomicity on the underlying probability space.

Remark 2.4.3 Let us note that, thanks to remark 1.2.4, the continuity assumption on G_Z in theorem 2.4.1 may be relaxed in the case of a capacity μ which, along with the properties required in theorem 2.4.1, has the additional property of continuity from above.
Let us further note that for a concave capacity μ (which is the case in theorem 2.4.1) the property of continuity from above of μ implies the property of continuity from below of μ.

2.4.1 The value function of problem (D) as a risk measure (or a premium principle in insurance)

In this subsection we study some of the properties of the value function $e(\cdot, Z)$ of problem (D), and we give an interpretation of $e(\cdot, Z)$ in terms of premium principles (or, up to a minus sign, in terms of risk measures). Our interpretation is analogous to that of Dana (2004).
So, let us consider the value function $e(\cdot, Z)$ of problem (D) for a fixed Z as a functional of the first argument and let us extend it to the whole set χ. More precisely, let us consider the functional $\rho : \chi \to \mathbb{R}$ defined by $\rho(X) := \rho^Z(X) := \int_0^1 r_Z(t) r_X(t) dt$ where Z is a fixed non-negative measurable function such that $\int_0^1 r_Z(t)dt < +\infty$. For the easing of the presentation, we will assume in the rest of this section that Z is such that $\int_0^1 r_{Z,\mu}(t)dt = 1$. This assumption is not a serious restriction because, due to the positive homogeneity of the objective functional of problem (D), we may as well replace Z by $\frac{Z}{\int_0^1 r_{Z,\mu}(t)dt}$ (in the case where $\int_0^1 r_{Z,\mu}(t)dt \neq 0$) in the formulation of problem (D).
The functional ρ is monotone ($X \leq Y$ implies $\rho(X) \leq \rho(Y)$) and translation invariant ($\rho(X+b) = \rho(X) + b, \forall b \in \mathbb{R}$). Therefore, according to the terminology used by Föllmer and Schied (2004), up to a minus sign, ρ is a monetary measure of risk on χ (see also Wang and Yan 2007, or Ekeland et al. 2009 for the same "sign convention" as the one used in the present paper). Moreover, ρ is additive with respect to comonotonic elements of χ ; this property is due to lemma 1.2.2. Monetary risk measures having the property of

comonotonic additivity have already been studied in the literature (cf. for instance Föllmer and Schied 2004), the idea being that when X and Y are comonotonic, X cannot act as a hedge against Y. The risk measure ρ has the additional property of being consistent with the increasing convex ordering relation $\leq_{icx,\mu}$, which means that if $X \leq_{icx,\mu} Y$ then $\rho(X) \leq \rho(Y)$. This consistency property is easily obtained thanks to proposition 2.3.5 applied with $g := r_Z$ (which is non-decreasing, non-negative and integrable).
We note furthermore that the functional ρ can be represented as a Choquet integral with respect to a capacity. Indeed, according to a well-known representation result for monotone and comonotonicly additive functionals on χ (cf. thm. 4.82. in Föllmer and Schied 2004, or Denneberg 1994), we know that there exists a capacity ν on $(\Omega, \mathcal{F})$ such that

$$\rho(X) = \mathbb{E}_{\nu}(X), \text{ for all } X \in \chi.$$

The capacity ν is related to the initial capacity μ in the following manner

$$\nu(A) = \rho(\mathbb{I}_A) = e(\mathbb{I}_A, Z) = \int_0^1 r_{Z,\mu}(t) r_{\mathbb{I}_A,\mu}(t)dt = \int_{1-\mu(A)}^1 r_{Z,\mu}(t)dt, \ \forall A \in \mathcal{F}.$$

Therefore, the capacity ν is of the form : $\nu(A) = \psi(\mu(A)), \ \forall A \in \mathcal{F}$ where $\psi(x) := \int_{1-x}^1 r_{Z,\mu}(t)dt, \forall x \in [0,1]$. We verify that the function ψ is a distortion function in the sense of the definition given in section 2.2 ; hence, the capacity $\nu = \psi \circ \mu$ is a distorted capacity. Moreover, the distortion function ψ being concave and the capacity μ being concave, the capacity ν is a concave capacity. Thus, the functional ρ can be represented as a Choquet integral with respect to the concave distorted capacity $\psi \circ \mu$; hence, ρ is a positively homogeneous, convex monetary risk measure (or, up to a minus sign, a coherent measure of risk in the terminology of Artzner et al. 1999).
Some of the previous observations are summarized in the following proposition for reader's convenience.

Proposition 2.4.2 *Let μ be a concave capacity. Let Z be a non-negative measurable function such that $\int_0^1 r_Z(t)dt = 1$. The functional $\rho : \chi \longrightarrow \mathbb{R}$ defined by $\rho(X) := \rho^Z(X) := \int_0^1 r_Z(t) r_X(t)dt$, for all $X \in \chi$, has the properties of monotonicity, translation invariance, comonotonic additivity, convexity and consistency with respect to the $\leq_{\mathrm{icx},\mu}$ $-$relation. Moreover, the functional ρ can be represented in the following manner :*

$$\rho(X) = \mathbb{E}_{\psi \circ \mu}(X), \forall X \in \chi$$

where ψ is a concave distortion function given by $\psi(x) := \psi^Z(x) := \int_{1-x}^1 r_Z(t)dt, \forall x \in [0,1]$.

Remark 2.4.4 We note that the assumption of concavity on the capacity μ of the previous proposition 2.4.2 is used only in the proof of the property of convexity of the functional ρ. The other statements of the proposition hold true without the assumption of concavity on μ.

As recalled in the introduction, risk measures of the form $\mathbb{E}_{\psi\circ\mu}(\cdot)$ where μ is a probability measure and ψ is a (concave) distortion function have been studied by Wang et al. (1997) and Denneberg (1990), and are now known under the name of distortion risk measures or distortion premium principles (see, for instance, Dhaene et al. 2006 for a survey and examples). At the end of his article, Denneberg (1990) suggests possible generalizations to the case where the probability measure is replaced by a more general set function : the functional ρ that we obtain could be seen as an example of such a generalization. Adopting this point of view, we could call ρ a "generalized" distortion risk measure.

Let us finally remark that the value function of problem (D) can also be seen as an analogue in the setting of ambiguity of the notion of maximal correlation risk measure (cf. Ekeland et al. 2009 and the references therein).

2.4.2 "Dual characterization" of the value function of problem (D)

In the previous subsection we have seen that the new premium principle $e(\cdot, Z)$ obtained by the insurance company through problem (D) has (among other "desirable" properties) the property of consistency with respect to the relation $\leq_{icx,\mu}$. We recall that the property of consistency with respect to a given stochastic dominance relation is often presented as a "desirable" property for a premium principle (cf. Young 2004 and the references therein, or Ruschendorf 2008). We note that a (consistent) premium principle induces a total pre-order on χ_+ (unlike the stochastic dominance relation which is only a partial pre-order).

In the following propostion we establish that the value function $e(\cdot, Z)$ of problem (D) is the smallest premium principle on χ_+ among those which are consistent with respect to the increasing convex dominance relation $\leq_{icx,\mu}$ and which are greater than or equal to the initial premium principle.

Proposition 2.4.3 ("dual characterization" of the value function of problem D) *The value function $e(\cdot, Z)$ of problem* (D) *is the smallest functional on χ_+ which satisfies the property of consistency with respect to the relation $\leq_{\text{icx},\mu}$ and which is greater than or equal to ρ_0, where ρ_0 is defined by $\rho_0(X) := \mathbb{E}_\mu(ZX)$, $\forall X \in \chi_+$.*

Proof: Let $X \in \chi_+$. We have $e(X, Z) \geq \mathbb{E}_\mu(ZX)$; this property is due to the fact that $e(\cdot, Z)$ is the value function of problem (D) and to the reflexivity of the relation $\leq_{icx,\mu}$.

Let $F : \chi_+ \mapsto \mathbb{R}$ be a functional which is consistent with $\leq_{icx,\mu}$ and which is greater than or equal to ρ_0. For all $C \in \chi_+$ such that $C \leq_{icx,\mu} X$, the property of consistency with respect to the relation $\leq_{icx,\mu}$ implies that $F(X) \geq F(C)$. Moreover, $F(C) \geq \mathbb{E}_\mu(ZC)$. So, by taking the supremum over the set $\{C \in \chi_+ \text{ s.t. } C \leq_{icx,\mu} X\}$, we have $F(X) \geq e(X, Z)$. $\square$

The previous proposition gives a link between the initial premium principle $\mathbb{E}_\mu(Z\cdot)$ and the "new" premium prinicple $e(\cdot, Z)$, which could be seen as analogous to that between the Value at Risk and the Average Value at Risk of theorem 9 in Kusuoka (2001) (cf. also theorem 4.61 in Föllmer and Schied 2004).
We end this section by recalling that in the review article by Young (2004) it is stressed on the importance of devising a premium principle according to a method in which the insurer "adopts a particular economic theory and then determines the resulting premium principle". In our case, the adopted economic theory is the CEU-theory, on which the definition (definition 2.3.1) of the $\leq_{icx,\mu}$ −relation is based. The $\leq_{icx,\mu}$ −relation is, in its turn, taken into account through the constraint of problem (D) in order to devise the new premium principle $e(\cdot, Z)$. Loosely speaking, the newly obtained premium principle is "richer" than the initial premium principle ρ_0 because other criteria of "riskiness" have been taken into account through the constraint of problem (D).

2.5 Subsequent work

In subsection 2.4.1 we have obtained that the value function $e(\cdot, Z)$ of problem (D) (where Z is fixed) can be represented as a Choquet integral with respect to a distorted capacity of the form $\psi \circ \mu$ where ψ is a concave distortion function. For a thorough study of functionals of the form $\mathbb{E}_{\psi\circ\mu}(\cdot)$ (where ψ is a distortion function and μ is a given capacity) the reader is referred to the following chapter. In particular, in theorem 3.4.4 of the following chapter it is established that under suitable assumptions on the underlying space $(\Omega, \mathcal{F}, \mu)$ any risk measure on χ_+ of the form $\mathbb{E}_{\psi\circ\mu}(\cdot)$ where ψ is a concave distortion function can be represented as a convex combination of the risk measure $\rho_\infty(\cdot)$, defined by $\rho_\infty(X) := \sup_{t<1} r_X^+(t)$, for all $X \in \chi_+$, and of a risk measure belonging to the family $\{e(\cdot, Z) : Z \geq 0 \text{ such that } \int_0^1 r_{Z,\mu}(t)dt = 1\}$.

2.A Some complements

The following result could be seen as a version of the monotone convergence theorem for Choquet integrals (theorem 1.2.1). The result is used in the proof of proposition 2.3.2 of section 2.3.

Proposition 2.A.1 *Let μ be a capacity, let (Z_n) be a* monotonic *sequence of real-valued measurable functions, and let Z denote the limit function. We assume that $\mathbb{E}_\mu(Z_0)$ exists and is finite, and $\mathbb{E}_\mu(Z)$ exists and is finite.*

1. *If μ is* continuous from below *and the sequence (Z_n) is* non-decreasing*, then the sequence $(\mathbb{E}_\mu(Z_n))$ converges* from below *to $\mathbb{E}_\mu(Z)$, i.e. $\mathbb{E}_\mu(Z_n) \uparrow \mathbb{E}_\mu(Z)$.*

2. *If μ is* continuous from above *and the sequence (Z_n) is* non-increasing*, then the sequence $(\mathbb{E}_\mu(Z_n))$ converges* from above *to $\mathbb{E}_\mu(Z)$, i.e. $\mathbb{E}_\mu(Z_n) \downarrow \mathbb{E}_\mu(Z)$.*

Roughly speaking, the previous proposition 2.A.1 states that the property of continuity from below (resp. from above) of a capacity μ is "transferred" to the Choquet integral with respect to μ. The proof of proposition 2.A.1 is based on the following lemma.

Lemma 2.A.1 *Let μ be a capacity which is continuous from below. Let (Z_n) be a non-decreasing sequence of real-valued measurable functions, and let Z denote the limit function.*

1. *The sequence of distribution functions (with respect to μ) of Z_n is non-increasing and converges to the distribution function (with respect to μ) of Z, i.e. $G_{Z_n}(x) \downarrow G_Z(x)$, for all $x \in \bar{\mathbb{R}}$.*
2. *The sequence $(r^-_{Z_n})$ of lower quantile functions is non-decreasing and converges to the lower quantile function r^-_Z of Z, i.e. $r^-_{Z_n}(t) \uparrow r^-_Z(t)$ for all $t \in (0,1)$.*
3. *The following convergence holds as well : $\quad r_{Z_n}(t) \uparrow r_Z(t)$ for almost every t, where r_{Z_n} and r_Z stand for (versions of) the quantile functions (with respect to μ) of Z_n and Z, respectively.*

The previous lemma 2.A.1 is a slight modification of lemma 1.4.1 of the previous chapter. Its proof is given for reader's convenience.

Proof of lemma 2.A.1 : The proof of the first statement is based on the same arguments as those of the proof of theorem 8.1 in Denneberg (1994).

We proceed to the proof of the second statement. As the sequence (Z_n) is non-decreasing, the sequence $(r^-_{Z_n})$ is non-decreasing; we denote by r its limit function, i.e. $r(t) := \lim_n r^-_{Z_n}(t) = \sup_n r^-_{Z_n}(t), \forall t \in (0,1)$. We will show that, for all $t \in (0,1), r(t) = r^-_Z(t)$.

Now, $G_{Z_n} \geq G_Z$ for all n, which implies that $r^-_{Z_n}(t) \leq r^-_Z(t), \forall t \in (0,1), \forall n$. By passing to the limit, we obtain $r(t) \leq r^-_Z(t), \forall t \in (0,1)$.

We turn to the proof of the converse inequality, namely $r(t) \geq r^-_Z(t), \forall t \in (0,1)$. Fix $t \in (0,1)$ and let $x \in \mathbb{R}$ be such that $G_Z(x) < t$. By the first part of the lemma, we know that $G_{Z_n}(x) \downarrow G_Z(x)$. Hence, there exists $n_0 = n_0(t,x)$ such that for all $n \geq n_0$, $G_{Z_n}(x) < t$. Therefore, for all $n \geq n_0$, $x \in \{y \in \mathbb{R} : G_{Z_n}(y) < t\}$ which implies that $r^-_{Z_n}(t) \geq x$, for all $n \geq n_0$. By passing to the limit, we obtain $r(t) \geq x$, which gives the desired inequality, and concludes the proof of the second statement.

The third statement of the lemma is a consequence of the second one, combined with the fact that $r^-_Z = r_Z$ almost everywhere and $r^-_{Z_n} = r_{Z_n}$ almost everywhere. □

Proof of proposition 2.A.1 : The proof of the first part of the proposition is based on the previous lemma 2.A.1, on the dominated convergence theorem for Lebesgue integrals applied to the sequence $(r_{Z_n}(\cdot))$, and on proposition 1.2.4. The dominated convergence theorem is applicable in virtue of the following three observations :

- Thanks to lemma 2.A.1, the sequence $(r_{Z_n}(\cdot))$ converges to $r_Z(\cdot)$ almost everywhere.
- For all $n \in \mathbb{N}$, $r_{Z_0}(\cdot) \leq r_{Z_n}(\cdot) \leq r_Z(\cdot)$ almost everywhere (due to the fact that, for all $n \in \mathbb{N}$, $Z_0 \leq Z_n \leq Z$).
- Thanks to the assumption on Z_0 and Z, and to proposition 1.2.4, the functions $r_{Z_0}(\cdot)$ and $r_Z(\cdot)$ are integrable.

By applying the dominated convergence theorem, we obtain $\int_0^1 r_{Z_n}(t)dt \uparrow \int_0^1 r_Z(t)dt$. We conclude thanks to proposition 1.2.4.

The second part of proposition 2.A.1 is a consequence of the first part applied with the sequence $(-Z_n)$ and with the (dual) capacity $\bar{\mu}$, and of the property of asymmetry of the Choquet integral. The details are straightforward, and are left to the reader. $\square$

2.B The proofs of Lemma 2.3.1 and Proposition 2.3.5

Proof of lemma 2.3.1 :

Throughout this proof we set $\phi(x) := G_X^{(2)}(x)$ to alleviate the notation. Accordingly, we denote by ϕ^* the conjugate function of ϕ, i.e. $\phi^*(y) := \sup_{x \in \mathbb{R}}(xy - \phi(x))$.

Let us first remark that

$$\phi(x) = \int_x^{+\infty} \mu(X > u)du = \mathbb{E}_\mu((X - x)_+) = \int_0^1 (r_X(t) - x)_+ dt, \tag{2.B.1}$$

where the second equality is the straightforward transformation used in the proof of proposition 2.3.3, and the third is due to proposition 1.2.4 and to lemma 1.2.1.

Thus, for $y = 0$, we have

$$\phi^*(0) = -\inf_{x \in \mathbb{R}} \int_0^1 (r_X(t) - x)_+ dt = -\lim_{x \to +\infty} \int_0^1 (r_X(t) - x)_+ dt = 0,$$

where we have used the non-increasingness of the function $x \mapsto \int_0^1 (r_X(t) - x)_+ dt$ and the Lebesgue convergence theorem. For $y = -1$, we have

$$\phi^*(-1) = -\lim_{x \to -\infty} \int_0^1 \max(r_X(t), x)dt = -\int_0^1 r_X(t)dt.$$

By analogous computations, we obtain that $\phi^*(y) = +\infty$ for $y > 0$, as well as $\phi^*(y) = +\infty$ for $y < -1$.

Finally, let us consider the case where $y \in (-1, 0)$.

The function f defined by $f(x) := xy - \phi(x)$ is concave (the function ϕ being convex). Noticing that $f(x) = xy - \int_x^{+\infty}(1 - G_X(u))du$, we get that the right-hand and left-hand derivatives of f at x are given by $f'_+(x) = y + (1 - G_X(x+))$ and $f'_-(x) = y + (1 - G_X(x-))$.

A point x is a maximum point for the function f if $\begin{cases} f'_+(x) \leq 0 \\ f'_-(x) \geq 0 \end{cases}$ which is equivalent to

$\begin{cases} G_X(x+) \geq y+1 \\ G_X(x-) \leq y+1 \end{cases}$ which, in turn, is equivalent to x being a $(y+1)$-quantile of X. By using this observation, the definition of ϕ^*, and equation (2.B.1), we have

$$\phi^*(y) = yr_X(y+1) - \int_0^1 (r_X(t) - r_X(y+1))_+ dt = -\int_{y+1}^1 r_X(t)dt,$$

which concludes the proof. □

Proof of Proposition 2.3.5 The implication (ii)⇒(i) is obtained by taking $g(t) := \mathbb{I}_{[y,1]}(t)$ which is non-negative, non-decreasing and integrable.
Let us now turn to the converse implication. Suppose that (i) holds true. The assertion (ii) is true for any function g of the form $g(t) := \mathbb{I}_{[y,1]}(t)$.
Let now g be a non-negative, non-decreasing step function. Then g can be written as follows : $g(t) = \sum_{i=1}^k a_i \mathbb{I}_{[bi,1]}(t)$, for almost every t, where $a_i \geq 0$ and $0 = b_1 < \cdots < b_k < 1$. Thus, we have

$$\int_0^1 g(t)r_X(t)dt = \sum_{i=1}^k a_i \int_{b_i}^1 r_X(t)dt \geq \sum_{i=1}^k a_i \int_{b_i}^1 r_Y(t)dt = \int_0^1 g(t)r_Y(t)dt.$$

Let now g be a non-negative, non-decreasing, integrable function. Then g can be approximated from below by a sequence (g_n) of non-negative, non-decreasing step functions. Due to the previous observation, we have $\int_0^1 g_n(t)r_X(t)dt \geq \int_0^1 g_n(t)r_Y(t)dt$. The function g being integrable, and the functions r_X and r_Y being bounded (since X and Y are in χ), we can apply the Lebesgue convergence theorem to pass to the limit in the previous inequality which concludes the proof. □

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CHAPITRE 3

Stochastic dominance with respect to a capacity and risk measures

Abstract : Pursuing our previous work in which the classical notion of *increasing convex* stochastic dominance relation with respect to a probability has been extended to the case of a normalized monotone (but not necessarily additive) set function also called a capacity, the present chapter gives a generalization to the case of a capacity of the classical notion of *increasing* stochastic dominance relation. This relation is characterized by using the notions of distribution function and quantile function with respect to the given capacity. Characterizations, involving Choquet integrals with respect to a distorted capacity, are established for the classes of monetary risk measures (defined on the space of bounded real-valued measurable functions) satisfying the properties of comonotonic additivity and consistency with respect to a given generalized stochastic dominance relation. Moreover, under suitable assumptions, a "Kusuoka-type" characterization is proved for the class of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the generalized increasing convex stochastic dominance relation. Generalizations to the case of a capacity of some well-known risk measures (such as the Value at Risk or the Tail Value at Risk) are provided as examples. It is also established that some well-known results about Choquet integrals with respect to a distorted probability do not necessarily hold true in the more general case of a distorted capacity.
Keywords : Choquet integral, stochastic orderings with respect to a capacity, distortion risk measure, quantile function with respect to a capacity, distorted capacity, Choquet expected utility, ambiguity, non-additive probability, Value at Risk, Rank-dependent expected utility, behavioural finance, maximal correlation risk measure, quantile-based risk measure, Kusuoka's characterization

3.1 Introduction

Capacities (which are normalized monotone set functions) and integration with respect to capacities were introduced by Choquet and were afterwards applied in different areas such as economics and finance among many others (cf. for instance Wang and Yan 2007 for an overview of applications). In economics and finance, capacities and Choquet integrals have been used, in particular, to build alternative theories to the "classical" setting of expected utility of Von Neumann and Morgenstern. Indeed, the classical expected utility paradigm has been challenged by various empirical experiments and "paradoxes" (such as Allais's and Ellsberg's) thus leading to the development of new theories. One of the proposed alternative theories is the Choquet expected utility, abridged as CEU, where agent's preferences are represented by a capacity μ and a non-decreasing real-valued function u. The agent's "satisfaction" with a claim X is then assessed by the Choquet integral of $u(X)$ with respect to the capacity μ. Choquet expected utility intervenes in situations where an objective probability measure is not given and where the agents are not able to derive a subjective probability over the set of different scenarios. Other alternative theories, such as the rank-dependent expected utility theory (Quiggin 1982) and Yaari's dual theory (Yaari 1997), can be seen as particular cases of the CEU-theory.

On the other hand, stochastic orders have also been extensively used in the decision theory. They represent partial order relations on the space of random variables on some probability space $(\Omega, \mathcal{F}, P)$ (more precisely, stochastic orders are partial order relations on the set of the corresponding distribution functions). Different kinds of stochastic orders, such as the *increasing* stochastic dominance (also known as first-order stochastic dominance) and the *increasing convex* stochastic dominance, have been studied and applied and links to the expected utility theory have been explored. The reader is referred to Müller and Stoyan (2002) and Shaked and Shanthikumar (2006) for a general presentation of the subject. As in the previous chapter the term "classical" will be used to designate the results in the case where the initial space $(\Omega, \mathcal{F}, P)$ is a *probability* space. We recall that a random variable X is said to be dominated by a random variable Y in the "classical" *increasing* (resp. ***inceasing convex***) stochastic dominance with respect to a given probability P if $\mathbb{E}_P(u(X)) \leq \mathbb{E}_P(u(Y))$ for all $u : \mathbb{R} \to \mathbb{R}$ *non-decreasing* (resp. ***non-decreasing and convex***) provided the expectations (taken in the Lebesgue sense) exist in $\mathbb{R}$. The definition of the "classical" stop-loss order, well-known in the insurance literature (cf., for instance, Denuit et al. 2006), is also recalled : X is said to be dominated by Y in the "classical" stop-loss order with respect to a given probability P if $\mathbb{E}_P((X-b)_+) \leq \mathbb{E}_P((Y-b)_+)$ for all $b \in \mathbb{R}$ provided the expectations (taken in the Lebesgue sense) exist in $\mathbb{R}$. We also recall that in the "classical" case of a probability the notions of increasing convex stochastic dominance and stop-loss order coincide.

In the previous chapter (cf. also Grigorova 2010), motivated by the CEU-theory, we have generalized the "classical" notion of *inceasing convex* stochastic dominance to the case where the measurable space $(\Omega, \mathcal{F})$ is endowed with a given capacity μ which is not necessarily a probability measure. It has been established in particular (cf. prop. 2.3.2 of chapter 2, or prop. 3.2 in Grigorova 2010) that the "classical" equivalence between the notions of increasing convex ordering and stop-loss ordering extends to the case where the capacity μ is assumed to be continuous from below and from above.

A closely related notion to the concepts mentioned above is the notion of risk measures having the properties of comonotonic additivity and consistency with respect to a given stochastic dominance relation. Risk measures having the property of consistency with respect to a given "classical" stochastic dominance relation have been extensively studied in the literature - cf. Dana (2005), Denuit et al. (2006), Song and Yan (2009 a.) and the references given by these authors. It is argued in Denuit et al. (2006) that "it seems reasonable to require that risk measures agree with some appropriate stochastic orders". On the other hand, risk measures having the property of comonotonic additivity have been introduced and links to the Choquet integrals have been explored (see, for instance, Schmeidler's representation theorem recalled in section 3.2 below). For the economic interpretation of the property of comonotonic additivity and further references the reader is referred to Föllmer and Schied (2004). Monetary risk measures having the properties of comonotonic additivity and consistency with respect to a given "classical" stochastic dominance relation have been linked to the so-called distortion risk measures, introduced in the insurance literature by Wang (1996) (cf. also Wang et al. 1997, as well as Dhaene et al. 2006 and references therein). Let us denote by χ the space of bounded real-valued measurable functions on $(\Omega, \mathcal{F})$ where $(\Omega, \mathcal{F})$ is a given measurable space. It is well-known (cf. the overview by Song and Yan 2009 c.) that the set of monetary risk measures defined on χ having the properties of comonotonic additivity and consistency with respect to the "classical" *increasing* stochastic dominance with respect to a given probability P can be characterized by means of Choquet integrals with respect to a capacity of the form $\psi \circ P$ where ψ is a distortion function (i.e. ψ is a non-decreasing function on $[0, 1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$). We recall that a capacity of the form $\psi \circ P$ where P is a probability and ψ is a distortion function is called a distorted probability. Under a non-atomicity assumption on the initial probability space $(\Omega, \mathcal{F}, P)$, the set of monetary risk measures defined on χ having the properties of comonotonic additivity and consistency with respect to the "classical" *stop-loss* stochastic dominance with respect to the probability P is known to be characterized by means of Choquet integrals with respect to a capacity of the form $\psi \circ P$ where ψ is a *concave* distortion function.

Moreover, some frequently used risk measures, such as the Value at Risk or the Tail Value at Risk among others, can be represented by means of Choquet integrals with respect to

a distorted probability (cf., for instance, Dhaene et al. 2006).
The notion of risk measures which are consistent with respect to a given "classical" stochastic dominance relation is also linked to the notion of law-invariance of risk measures introduced by Kusuoka (2001). Kusuoka (2001) has provided a characterization of the class of convex law-invariant comonotonicly additive monetary risk measures on the space $L^{\infty}(\Omega, \mathcal{F}, P)$ in the case where the probability space $(\Omega, \mathcal{F}, P)$ is atomless (cf. theorem 7 in Kusuoka 2001, as well as theorem 1.4 in Ekeland and Schachermayer 2011).

In the present chapter we pursue our previous work from the previous chapter (cf. also Grigorova 2010) by generalizing the "classical" notion of *increasing* stochastic dominance to the case where the measurable space $(\Omega, \mathcal{F})$ is endowed with a capacity μ which is not necessarily a probability measure. We characterize this "generalized" relation by using the notions of distribution function with respect to the capacity μ and quantile function with respect to the capacity μ. Next, we study the set of monetary risk measures defined on χ having the properties of comonotonic additivity and consistency with respect to the *"generalized" increasing stochastic dominance* with respect to the capacity μ, as well as the set of monetary risk measures defined on χ having the properties of comonotonic additivity and consistency with respect to the *"generalized" stop-loss stochastic dominance* with respect to the capacity μ. Under suitable assumptions on the space $(\Omega, \mathcal{F}, \mu)$ we provide characterizations analogous to the classical ones. More precisely, in the case where the initial capacity μ is assumed to be continuous from below and from above, the former class of risk measures is characterized in terms of Choquet integrals with respect to a capacity of the form $\psi \circ \mu$ (which we call a distorted capacity) where ψ is a distortion function. Under suitable assumptions on the space $(\Omega, \mathcal{F}, \mu)$ the latter class of risk measures is characterized by means of Choquet integrals with respect to a distorted capacity of the form $\psi \circ \mu$ whose distortion function ψ is concave. We also establish that some well-known results concerning Choquet integrals with respect to a distorted probability do not necessarily hold true in the more general case of a distorted capacity (cf. subsection 3.3.4, as well as remarks 3.4.1 and 3.4.2). After reformulating Kusuoka's theorem in a form which is suitable for the needs of the present chapter, we establish a "Kusuoka-type" characterization of the class of monetary risk measures defined on χ having the properties of comonotonic additivity and consistency with respect to the *"generalized" stop-loss stochastic dominance* with respect to the capacity μ. According to this characterization (cf. theorem 3.4.3 below) the risk measures ρ_∞ and ρ^Y defined by $\rho_\infty(X) := \sup_{t<1} r^+_{X,\mu}(t)$ for all $X \in \chi$ and $\rho^Y(X) := \int_0^1 r^+_{Y,\mu}(t) r^+_{X,\mu}(t)dt$ for all $X \in \chi$, where Y is a non-negative measurable function on $(\Omega, \mathcal{F})$ such that $\int_0^1 r^+_{Y,\mu}(t)dt = 1$, can be viewed as the "building blocks" of the latter class of risk measures [1]. Under additional assumptions on the initial capacity μ (namely continuity from below and from above, and concavity) a characterization involving the value function of the optimization problem (D) studied in the previous chapter (cf. also Grigorova 2010) is given in theorem 3.4.4. We end this chapter by giving some examples generalizing the "classical" Value at Risk and the "classical" Tail Value at Risk to the case of a capacity which is not necessarily a probability measure. In the case of the "generalized" Value at Risk some particular subcases are studied and an economic interpretation is provided.

The present chapter is based on our working paper Grigorova (2011) : "Stochastic dominance with respect to a capacity and risk measures", hal-00639667.
The remainder of the chapter is organized in the following manner. Section 3.2 is divided in two subsections. Subsection 3.2.1 recalls (from the previous chapter 2) the definitions and characterizations of the "generalized" increasing convex ordering and the "generalized" stop-loss ordering with respect to a capacity in a form which is suitable for the needs of the present chapter ; the proofs of the results of this subsection can be found in the previous chapter (or in Grigorova 2010). The terminology about risk measures is recalled in subsection 3.2.2, along with a useful representation result due to D. Schmeidler.
Section 3.3 is divided in four subsections. In subsection 3.3.1 we define the "generalized" increasing stochastic ordering with respect to a capacity and provide characterizations analogous to those in the classical case of a probability measure. In subsection 3.3.2 we characterize the set of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the "generalized" increasing stochastic ordering (with respect to a given capacity μ). Subsection 3.3.3 is devoted to the characterization of the set of monetary risk measures which are comonotonicly additive and consistent with respect to the "generalized" stop-loss stochastic ordering (with respect to a given capacity μ). Subsection 3.3.4 deals with the property of convexity of a Choquet integral with respect to a distorted capacity of the form $\psi \circ \mu$.
In section 3.4 (theorem 3.4.3 and theorem 3.4.4) we provide "Kusuoka-type" characterizations of the set of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the "generalized" stop-loss stochastic ordering (with respect to a given capacity μ).
Section 3.5 is devoted to the examples.
The appendix is divided in three parts : the first part contains some proofs ; the second part contains an observation on one of the representation results from section 3.3, namely on lemma 3.3.1 ; the third part is devoted to detailed explanations concerning remark 3.3.4

1. The symbol $r^{+}_{X,\mu}$ (resp. $r^{+}_{Y,\mu}$) denotes the upper quantile function of X (resp. of Y) with respect to the capacity μ ; the reader is referred to chapter 1 for more details on quantile functions.

from section 3.3.

3.2 Some basic definitions and results

3.2.1 Stochastic orderings with respect to a capacity

In this subsection we summarize some of the results on the "generalized" stochastic dominance relation of the previous chapter in a form which suits our present purpose. The notation and definitions on capacities and Choquet integrals are the same as those of the previous chapters.

Definition 3.2.1 *Let X and Y be two real-valued measurable functions on $(\Omega, \mathcal{F})$ and let μ be a capacity on $(\Omega, \mathcal{F})$. We say that X is smaller than Y in the* increasing convex *ordering (with respect to the capacity μ), denoted by $X \leq_{icx} Y$, if*

$$\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$$

for all functions $u : \mathbb{R} \to \mathbb{R}$ which are non-decreasing and convex, provided the Choquet integrals exist in $\mathbb{R}$.

We define the notion of stop-loss ordering (or stop-loss dominance relation) below.

Definition 3.2.2 *Let X and Y be two real-valued measurable functions on $(\Omega, \mathcal{F})$ and let μ be a capacity on $(\Omega, \mathcal{F})$. We say that X is smaller than Y in the* stop-loss *ordering with respect to the capacity μ, denoted by $X \leq_{\mathrm{sl}} Y$, if*

$$\mathbb{E}_\mu((X-b)_+) \leq \mathbb{E}_\mu((Y-b)_+), \ \forall b \in \mathbb{R},$$

provided the Choquet integrals exist in $\mathbb{R}$.

In the classical case where the capacity μ is a probability measure the previous definition is reduced to the usual definition of stop-loss order well-known in the insurance literature (see for instance Dhaene et al. 2006). The interpretation of the stop-loss dominance relation in the classical case is the following : $X \leq_{sl} Y$ if and only if X has lower stop-loss premia than Y. A similar interpretation could be given in our more general setting if we see the number $\mathbb{E}_\mu((X-b)_+)$ for a given $b \in \mathbb{R}$ as a "generalized" stop-loss premium of X.

The following characterization of the stop-loss ordering relation with respect to a capacity is due to propositions 2.3.3 and 2.3.4 in chapter 2 (see also propositions 3.3 and 3.4 in Grigorova 2010).

Proposition 3.2.1 *Let μ be a capacity and let X and Y be two real-valued measurable functions such that $\int_0^1 |r_X(t)|dt < +\infty$ and $\int_0^1 |r_Y(t)|dt < +\infty$ where r_X and r_Y denote (the) quantile functions of X and Y with respect to μ. The following three statements are equivalent :*

(i) $X \leq_{\mathrm{sl},\mu} Y$.

(ii) $\int_x^{+\infty} \mu(X > u)du \leq \int_x^{+\infty} \mu(Y > u)du$, $\forall x \in \mathbb{R}$.

(iii) $\int_y^1 r_X(t)dt \leq \int_y^1 r_Y(t)dt, \forall y \in (0,1)$.

Another useful characterization of the relation $\leq_{sl,\mu}$ is given in the following proposition ; its proof can be found in the previous chapter.

Proposition 3.2.2 *Let $X \in \chi$ and $Y \in \chi$ be given. Then the following statements are equivalent :*

(i) $X \leq_{\mathrm{sl},\mu} Y$

(ii) $\int_0^1 g(t)r_X(t)dt \leq \int_0^1 g(t)r_Y(t)dt$, $\forall g : [0,1] \to \bar{\mathbb{R}}_+$, *integrable, non-decreasing.*

We have the following proposition establishing the equivalence between the increasing convex stochastic dominance and the stop-loss stochastic dominance in the case of a capacity which is continuous from below and from above (cf. proposition 2.3.2 of chapter 2, or proposition 3.2 in Grigorova 2010).

Proposition 3.2.3 *Let μ be a capacity which is continuous from below and from above and let X and Y be two real-valued measurable functions. Then the following two statements are equivalent :*

(i) $X \leq_{\mathrm{sl},\mu} Y$.

(ii) $X \leq_{\mathrm{icx},\mu} Y$.

Remark 3.2.1 As observed in the previous chapter, it can be easily seen from the definition of the increasing convex ordering that the assumption on the continuity of the capacity μ is not needed in the proof of the implication $(ii) \Rightarrow (i)$ in proposition 3.2.3.

3.2.2 Monetary risk measures

We will use the following definitions :

Definition 3.2.3 *1. A mapping $\rho : \chi \to \mathbb{R}$ is called a monetary measure of risk if it satisfies the following properties for all $X, Y \in \chi$:*

(i) *(monotonicity)* $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$

(ii) *(translation invariance)* $\rho(X + b) = \rho(X) + b, \forall b \in \mathbb{R}$

2. *A monetary measure of risk ρ is called convex if it satisfies the additional property of*

   (iii) *(convexity)* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, $\forall \lambda \in [0,1]$, $\forall X, Y \in \chi$.

3. *A convex monetary measure of risk ρ is called coherent if it satisfies the additional property of*

   (iv) *(positive homogeneity)* $\rho(\lambda X) = \lambda\rho(X), \forall \lambda \in \mathbb{R}_+$.

Let us remark that the above definiton of a coherent monetary risk measure coincides, up to a minus sign, with the definiton given by Artzner et al. (1999). The "sign convention" which we use is frequently adopted in the insurance literature when the measurable functions are interpreted as potential losses or payments that have to be made (see, for instance, Dhaene et al. 2006 for explanations in the context of insurance; for the same "sign convention" as the one used in the present chapter, the reader is also referred to Wang and Yan 2007, or Ekeland et al. 2009).

The next theorem is known as Schmeidler's representation theorem (cf. theorem 11.2 in Denneberg 1994; cf. also theorem 4.82 in Föllmer and Schied 2004).

Theorem 3.2.1 (Schmeidler's representation theorem) *Let $\rho : \chi \longrightarrow \mathbb{R}$ be a given functional satisfying the properties of :*

(i) *(monotonicity)* $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$

(ii) *(comonotonic additivity)* X, Y *comonotonic* $\Rightarrow \rho(X+Y) = \rho(X) + \rho(Y)$

(iii) *(normalization)* $\rho(\mathbb{I}) = 1$.

Then, there exists a capacity ν on $(\Omega, \mathcal{F})$ such that

$$\rho(X) = \mathbb{E}_\nu(X),\ \forall X \in \chi.$$

Remark 3.2.2 We note that the normalization property (iii) of the previous theorem is satisfied by any functional $\rho : \chi \longrightarrow \mathbb{R}$ which is assumed to have the properties of comonotonic additivity (property (ii)) and translation invariance. Indeed, the comonotonic additivity of ρ implies that $\rho(0+0) = 2\rho(0)$ which gives $\rho(0) = 0$. This property combined with the translation invariance of ρ implies the normalization property (iii). In particular, the normalization property (iii) is satisfied by any monetary risk measure $\rho : \chi \longrightarrow \mathbb{R}$ (in the sense of definition 3.2.3) having the property of comonotonic additivity.

3.3 Stochastic orderings with respect to a capacity and generalized distortion risk measures

3.3.1 The increasing stochastic dominance with respect to a capacity

In this subsection we define the notion of increasing stochastic dominance with respect to a capacity and provide characterizations analogous to those existing in the "classical" case of a probability measure. The reader is referred to Shaked and Shanthikumar (2006) for details in the classical case.

Definition 3.3.1 *Let X and Y be two real-valued measurable functions on $(\Omega, \mathcal{F})$ and let μ be a capacity on $(\Omega, \mathcal{F})$. We say that X is dominated by Y in the* increasing *stochastic dominance (with respect to the capacity μ), denoted by $X \leq_{\text{mon},\mu} Y$, if*

$$\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$$

for all non-decreasing functions $u : \mathbb{R} \to \mathbb{R}$ provided the Choquet integrals exist in $\mathbb{R}$.

In the case where μ is a probability measure the preceding definition is reduced to the usual definition of increasing stochastic dominance (also known as first-order stochastic dominance).

Remark 3.3.1 The economic interpretation of the increasing stochastic dominance with respect to a capacity μ is the following : $X \leq_{mon,\mu} Y$ if all the CEU-maximizers whose preferences are described by the (common) capacity μ and a non-decreasing utility function (and who associate a real number to their satisfaction with X and Y) prefer the claim Y to the claim X.

The idea of defining an increasing (or decreasing) stochastic dominance relation in the case of a capacity is already present in a paper by Scarsini (1992). Some comments on the links between the notion studied in the present subsection and the work by Scarsini (1992) are made in remark 3.3.4.
We have the following characterization of the increasing stochastic dominance with respect to μ.

Proposition 3.3.1 *Let μ be a capacity which is continuous from below and from above. Let X and Y be two real-valued measurable functions. The following three statements are equivalent :*

(i) $X \leq_{\text{mon},\mu} Y$.

(ii) $G_X(x) \geq G_Y(x)$, $\forall x \in \mathbb{R}$.

(iii) $r_X^+(t) \leq r_Y^+(t), \forall t \in (0,1)$.

Proof : Let us first prove the implication $(i) \Rightarrow (ii)$. We fix $x \in \mathbb{R}$ and we remark that $G_X(x) = 1 - \mathbb{E}_\mu(u(X))$ where $u(y) := \mathbb{I}_{(x,+\infty)}(y)$ which proves the desired implication as the function u is non-decreasing.
The implication $(ii) \Rightarrow (iii)$ is a consequence of the definition of the upper quantile functions r_X^+ and r_Y^+.
To conclude, we prove the implication $(iii) \Rightarrow (i)$. Suppose that $r_X^+(t) \leq r_Y^+(t), \forall t \in (0,1)$ and let $u : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Thanks to proposition 1.2.4 and to remark 1.2.4 (where the assumption of continuity from below and from above of μ is used) we have $\mathbb{E}_\mu(u(X)) = \int_0^1 u(r_X^+(t))dt$; the same type of representation holds for $\mathbb{E}_\mu(u(Y))$. Thus we obtain $\mathbb{E}_\mu(u(X)) = \int_0^1 u(r_X^+(t))dt \leq \int_0^1 u(r_Y^+(t))dt = \mathbb{E}_\mu(u(Y))$ which concludes the proof.

□

Remark 3.3.2 We note that the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ in the proof of proposition 3.3.1 have been established without using the assumption of continuity from below and from above of μ.

Remark 3.3.3 If the capacity μ is not continuous, the implication $(ii) \Rightarrow (i)$ of the previous proposition 3.3.1 may not hold true. The following counter-example is inspired by Denneberg (1994) (cf. exercise 4.1).
We set $\Omega := \mathbb{R}$ and $\mathcal{F} := \mathcal{P}(\mathbb{R})$. We consider the set function μ on $(\Omega, \mathcal{F})$ defined in the following manner : for $A \in \mathcal{F}$,

$$\mu(A) := \begin{cases} 1, \text{ if there exist } \varepsilon^- < 0 < \varepsilon^+ \text{ such that }]\varepsilon^-, \varepsilon^+[\subset A \\ 0, \text{ otherwise.} \end{cases}$$

The set function μ is a capacity in the sense of definition 1.2.1. Moreover, we can check that μ is neither continuous from below, nor continuous from above. Indeed, we can see that μ is not continuous from below by considering the *non-decreasing* sequence of measurable sets (A_n) defined by $A_n := (-\infty, 0] \cup [\frac{1}{n}, +\infty)$, for all $n \in \mathbb{N}^*$, and by noting that $\cup_n A_n = \mathbb{R}$, $\mu(A_n) = 0$, for all $n \in \mathbb{N}^*$, and $\mu(\cup_n A_n) = 1$. We can see that μ is not continuous from above by considering the *non-increasing* sequence of measurable sets (B_n) defined by $B_n := (-\frac{1}{n}, +\infty)$, for all $n \in \mathbb{N}^*$. We observe that $\cap_n B_n = [0, +\infty)$, $\mu(B_n) = 1$, for all $n \in \mathbb{N}^*$, and $\mu(\cap_n B_n) = 0$.
We will exhibit two measurable functions X and Y on $(\Omega, \mathcal{F})$, and a non-decreasing function u, such that $G_{X,\mu} \leq G_{Y,\mu}$ and $\mathbb{E}_\mu(u(X)) < \mathbb{E}_\mu(u(Y))$, which will give the desired counter-example.
Let us consider the following two measurable functions X and Y on $(\Omega, \mathcal{F})$: $Y \equiv 0$ (i.e. Y is identically equal to zero) and $X = id$ (i.e. X is the identity function). An ex-

plicit computation gives $G_{X,\mu}(x) = G_{Y,\mu}(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$. In particular, we have $G_{X,\mu} \leq G_{Y,\mu}$. Let us consider the non-decreasing function $u : \mathbb{R} \to \mathbb{R}$ defined by $u(x) := a\mathbb{I}_{[0,+\infty)}(x) + b\mathbb{I}_{(0,+\infty)}(x) - a$, for all $x \in \mathbb{R}$, where $a > 0$ and $b > 0$. We have $\mathbb{E}_\mu(u(Y)) = \mathbb{E}_\mu(u(0)) = \mathbb{E}_\mu(0) = 0$. We have $\mathbb{E}_\mu(u(X)) = a\mu([0, +\infty)) + b\mu((0, +\infty)) - a = -a$, where we have used the properties of translation invariance, comonotonic additivity and positive homogeneity of the Choquet integral to obtain the first equality.
The measurable functions X and Y satisfy the property $G_{X,\mu} \leq G_{Y,\mu}$ (property (ii) of the previous proposition 3.3.1). Nevertheless, Y is not dominated by X in the sense of the $\leq_{mon,\mu}$ −relation as $\mathbb{E}_\mu(u(X)) < \mathbb{E}_\mu(u(Y))$.

Remark 3.3.4 The reader who is familiar with the paper by Scarsini (1992) could observe that the equivalence between assertions (i) and (ii) in the above proposition 3.3.1 could be deduced from theorem 2.2 part (a) in Scarsini (1992) (applied with the "image capacities" $\bar{\mu} \circ X^{-1}$ and $\bar{\mu} \circ Y^{-1}$). However, let us draw the reader's attention to the following : in theorem 2.2 of Scarsini (1992) there is not any assumption of continuity on the capacities ; on the other hand, we have seen (cf. the previous remark 3.3.3) that without such an assumption the equivalence between (i) and (ii) in proposition 3.3.1 may not hold true. In fact, theorem 2.2 part (a) of Scarsini (1992), such as stated by the author, may not hold true : the previous remark 3.3.3 gives a counter-example. Some more details are given in the appendix 3.C.

We end this subsection by giving some vocabulary which will be useful in the sequel while dealing with risk measures.

Definition 3.3.2 *Let $\rho : \chi \to \mathbb{R}$ be a given functional and let μ be a capacity. We say that ρ satisfies the property of :*

1. *(*consistency with respect to $\leq_{\text{mon},\mu}$*) if $X \leq_{\text{mon},\mu} Y$ implies $\rho(X) \leq \rho(Y)$.*
2. *(*consistency with respect to $\leq_{\text{sl},\mu}$*) if $X \leq_{\text{sl},\mu} Y$ implies $\rho(X) \leq \rho(Y)$.*
3. *(*consistency with respect to $\leq_{\text{icx},\mu}$*) if $X \leq_{\text{icx},\mu} Y$ implies $\rho(X) \leq \rho(Y)$.*

The following result, which is easy to establish, provides a link between the notions introduced in definition 3.3.2.

Proposition 3.3.2 *Let $\rho : \chi \to \mathbb{R}$ be a given functional and let μ be a capacity. The following statements hold :*

1. *ρ is consistent with respect to $\leq_{\text{sl},\mu} \Rightarrow$ ρ is consistent with respect to $\leq_{\text{icx},\mu} \Rightarrow$ ρ is consistent with respect to $\leq_{\text{mon},\mu}$.*

2. *If the capacity μ is continuous from below and from above, the consistency with respect to the relation $\leq_{\mathrm{icx},\mu}$ is equivalent to the consistency with respect to the relation $\leq_{\mathrm{sl},\mu}$.*

Proof: The first statement is due to the definitions of the relations $\leq_{icx,\mu}$, $\leq_{sl,\mu}$ and $\leq_{mon,\mu}$. The second statement is a consequence of proposition 3.2.3. □

3.3.2 Generalized distortion risk measures

In this subsection we are interested in risk measures which can be represented as Choquet integrals with respect to a distorted capacity. Such risk measures will be called generalized distortion risk measures.

Definition 3.3.3 *Let μ be a capacity. A monetary risk measure $\rho : \chi \longrightarrow \mathbb{R}$ of the form*

$$\rho(X) = \mathbb{E}_{\psi \circ \mu}(X), \ \forall X \in \chi, \ \textit{where } \psi \textit{ is a distortion function,}$$

is called a generalized distortion risk measure with respect to μ.

In the case where μ is a probability measure the previous definition is reduced to the definition of a distortion risk measure (or a distortion premium principle) well-known in finance and insurance - see, for instance, Dhaene et al. (2006) for a survey and examples. The generalization considered in definition 3.3.3 is suggested at the end of an article by Denneberg (1990). In the previous chapter (cf. also Grigorova 2010) an example of a generalized distortion risk measure is obtained as the value function of the following financial optimization problem :

$$\begin{gathered}\text{Maximize } \mathbb{E}_\mu(ZC) \\ \text{under the constraints } C \in \chi_+ \text{ s.t. } C \leq_{icx,\mu} X\end{gathered} \tag{D}$$

where χ_+ denotes the set of non-negative bounded measurable functions, μ is a given (concave and continuous from below) capacity, Z is a given non-negative measurable function such that $\int_0^1 r_Z(t)dt < \infty$ and X is a given function in χ_+.

Remark 3.3.5 Any generalized distortion risk measure in the sense of definition 3.3.3 is a monetary risk measure satisfying the properties of positive homogeneity and comonotonic additivity. A generalized distortion risk measure is convex if and only if the distorted capacity $\psi \circ \mu$ appearing in definition 3.3.3 is a concave capacity. The "if part" in the previous statement has already been recalled in proposition 1.2.3 ; the "only if part" is easy to establish by using, for instance, exercise 5.1 in Denneberg (1994).

A representation result well-known in the classical case of a probability measure is generalized to the case of a capacity in the following lemma. For the statement and the proof of this result in the "classical" case we refer to Song and Yan (2009 a.) as well as to exercise 11.3 in Denneberg (1994) ; the "classical" result is related to the work of Wang et al. (1997) and to the work of Kusuoka (2001) as well.

Lemma 3.3.1 *Let μ be a capacity on $(\Omega, \mathcal{F})$ and let $\rho : \chi \to \mathbb{R}$ be a functional satisfying the following properties*

(i) $G_{X,\mu}(x) \geq G_{Y,\mu}(x), \forall x \in \mathbb{R} \Rightarrow \rho(X) \leq \rho(Y)$

(ii) *(comonotonic additivity)* X, Y *comonotonic* $\Rightarrow \rho(X+Y) = \rho(X) + \rho(Y)$

(iii) *(normalization)* $\rho(\mathbb{I}) = 1.$

Then, there exists a distortion function $\psi : [0,1] \longrightarrow [0,1]$ *such that*

$$\rho(X) = \mathbb{E}_{\psi \circ \mu}(X),\ \forall X \in \chi.$$

This lemma is based on Schmeidler's representation theorem (theorem 3.2.1). Before we prove the lemma, let us make a remark which will be used in the proof.

Remark 3.3.6 The property (i) in the previous lemma implies the property of monotonicity of ρ (i.e. $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$), as well as the following property which, for the easing of the presentation, will be called *distribution invariance of ρ with respect to μ* :

$$G_{X,\mu}(x) = G_{Y,\mu}(x),\ \forall x \in \mathbb{R} \Rightarrow \rho(X) = \rho(Y).$$

Proof of lemma 3.3.1 : The functional ρ being monotonic, comonotonic additive and normalized, Schmeidler's representation theorem (theorem 3.2.1) can be applied in order to obtain the existence of a capacity ν on $(\Omega, \mathcal{F})$ such that

$$\rho(X) = \mathbb{E}_{\nu}(X),\ \forall X \in \chi. \tag{3.3.1}$$

We will now prove that there exists a distortion function ψ such that $\nu(A) = \psi \circ \mu(A),\ \forall A \in \mathcal{F}$. The arguments are similar to those in the "classical" case and follow the proof of proposition 2.1 in Song and Yan (2009 a.).

Let us first note that for $A, B \in \mathcal{F}$, the distribution functions (with respect to μ) $G_{\mathbb{I}_A,\mu}$ and $G_{\mathbb{I}_B,\mu}$ of the measurable functions $\mathbb{I}_A$ and $\mathbb{I}_B$ coincide if and only if $\mu(A) = \mu(B)$. Thus, the functional ρ being distribution invariant with respect to μ, we have that $\mu(A) = \mu(B)$ implies $\rho(\mathbb{I}_A) = \rho(\mathbb{I}_B)$ which in turn implies that $\nu(A) = \nu(B)$. Therefore, we can define a function ψ on the set $S := \{\mu(A), A \in \mathcal{F}\}$ as follows :

$$\begin{aligned} \psi : \{\mu(A), A \in \mathcal{F}\} &\longrightarrow [0,1] \\ \psi(x) &:= \nu(A) \text{ if } x = \mu(A). \end{aligned}$$

The function ψ is such that $\nu(A) = \psi \circ \mu(A)$, $\forall A \in \mathcal{F}$. Moreover, $\psi(0) = 0$ and $\psi(1) = 1$ and ψ is a non-decreasing function on S. The non-decreasingness of ψ is a consequence of property (i). Indeed, let $A, B \in \mathcal{F}$ be such that $\mu(A) \leq \mu(B)$. Then, for all $x \in \mathbb{R}$, $G_{\mathbb{I}_A,\mu}(x) = 1 - \mu(\mathbb{I}_A > x) \geq 1 - \mu(\mathbb{I}_B > x) = G_{\mathbb{I}_B,\mu}(x)$. The inequality $\nu(A) \leq \nu(B)$ follows thanks to property (i) and to the representation (3.3.1). We conclude the proof as in Song and Yan (2009 a.) by arguing that the function ψ can be extended to a non-decreasing function on the closure of the set S and then to a non-decreasing function on $[0, 1]$.

□

Remark 3.3.7 The converse statement in lemma 3.3.1 also holds true. More precisely, let μ be a capacity and let $\rho : \chi \to \mathbb{R}$ be a functional of the form $\rho(\cdot) = \mathbb{E}_{\psi\circ\mu}(\cdot)$ where ψ is a distortion function. As a Choquet integral with respect to a capacity, the functional ρ obviously satisfies properties (ii) and (iii) in lemma 3.3.1. Property (i) in lemma 3.3.1 is also satisfied as the functional ρ can be written in the following manner : $\rho(X) = \mathbb{E}_{\psi\circ\mu}(X) = \int_0^{+\infty} \psi(1 - G_{X,\mu}(x))dx + \int_{-\infty}^0 \psi(1 - G_{X,\mu}(x)) - 1dx$, $\forall X \in \chi$.

Some additional observations on the above lemma 3.3.1 are made in the appendix 3.B.

The following theorem is a "generalization" to the case of a capacity of a well-known representation result for monetary risk measures satisfying the properties of comonotonic additivity and consistency with respect to the "classical" increasing stochastic dominance (see for instance Song and Yan 2009 a. for the classical case).

Theorem 3.3.1 *Let μ be a capacity on $(\Omega, \mathcal{F})$ which is continuous from below and from above and let $\rho : \chi \to \mathbb{R}$ be a monetary risk measure satisfying the properties of*

(i) *(consistency with respect to $\leq_{\text{mon},\mu}$) $X \leq_{\text{mon},\mu} Y \Rightarrow \rho(X) \leq \rho(Y)$*

(ii) *(comonotonic additivity) X, Y comonotonic $\Rightarrow \rho(X + Y) = \rho(X) + \rho(Y)$.*

Then, there exists a distortion function ψ such that

$$\rho(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi.$$

Proof : The result follows directly from lemma 3.3.1 and proposition 3.3.1.

□

Remark 3.3.8 Note that properties (i) and (ii) in the previous theorem are satisfied by any monetary risk measure on χ of the form $\mathbb{E}_{\psi\circ\mu}(\cdot)$ where ψ is a given distortion function and where μ is a given capacity. The statement is due to remark 3.3.7, to proposition 3.3.1 and to remark 3.3.2.

Remark 3.3.9 We also note that the distortion function ψ in the representation formula of the previous theorem is unique on the set $S := \{\mu(A), A \in \mathcal{F}\}$.

We conclude from the previous theorem 3.3.1 combined with remark 3.3.8 that in the case where the initial capacity μ is continuous from below and from above the class of generalized distortion risk measures with respect to μ (in the sense of definition 3.3.3) coincides with the class of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{mon,\mu}$ −relation.

As already mentioned, risk measures satisfying the property of comonotonic additivity (property (ii) in the previous theorem) have been extensively studied in the literature and the financial interpretation of this property has been aknowledged (see for instance Föllmer and Schied 2004).

We give hereafter an interpretation of the notion of consistency with respect to a given "generalized" stochastic dominance relation. The interpretation provided in this chapter is from the point of view of an insurance company. Consider an insurance company which is willing to compare measurable functions (interpreted in this context as random losses) according to the CEU-theory. The use of a stochastic dominance relation deriving from the CEU-theory (such as the $\leq_{mon,\mu}$ - stochastic dominance relation, the $\leq_{sl,\mu}$ −relation or the $\leq_{icx,\mu}$ −relation) is suitable as it gives a way of comparing random losses according to the desired economic theory. The CEU-theory and the stochastic dominance relations to which it gives rise may intervene, for instance, in situations where the insurance company is facing ambiguity. However, as is the case of the "classical" stochastic dominance relations with respect to a probability, the stochastic dominance relations with respect to a capacity have the following "drawback" : the relations are not "total" which means that for some measurable functions X and Y it is possible to have neither $X \leq_{mon,\mu} Y$ nor $Y \leq_{mon,\mu} X$ (if the $\leq_{mon,\mu}$ −relation is taken as an example).

In the present chapter risk measures having the property of consistency with respect to the given stochastic dominance relation with respect to a capacity are used as a way of circumventing the previous "drawback". This approach is analogous to the one used in the "classical" case of a probability where risk measures consistent with respect to the "classical" stochastic dominance relations are studied.

Remark 3.3.10 The $\leq_{mon,\mu}$ −relation and the property of consistency with respect to the $\leq_{mon,\mu}$ −relation could be interpreted in terms of ambiguity. The interpretation is based on the characterization of the $\leq_{mon,\mu}$ −relation established in proposition 3.3.1 in the case of a capacity μ which is continuous from below and from above. Let us recall that in the present chapter the measurable functions on $(\Omega, \mathcal{F})$ are interpreted as losses, and let X and Y be two measurable functions in χ such that

$$G_{X,\mu}(t) \leq G_{Y,\mu}(t) \text{ for all } t \in \mathbb{R} \tag{3.3.2}$$

which is equivalent to $\mu(X > t) \geq \mu(Y > t)$ for all $t \in \mathbb{R}$.
Let us first consider the inequality $\mu(X > t) \geq \mu(Y > t)$ where $t \in \mathbb{R}$ is fixed. Bearing in mind that the capacity μ models the agent's perception of "uncertain" (or "ambiguous") events, the reader may interpret the previous inequality as having the following meaning : the event $\{X > t\}$ is perceived by the agent as being less uncertain than (or equally uncertain to) the event $\{Y > t\}$.
Thus, the relation (3.3.2) (which, thanks to proposition 3.3.1, is equivalent to $Y \leq_{mon,\mu} X$ in the case of a capacity μ assumed to be continuous from below and from above) can be loosely read as follows : the agent "feels less (or equally) uncertain about the loss X's taking great values than about the loss Y's ".
Thus, if a loss $X \in \chi$ is perceived (through a capacity μ which is continuous from below and from above) as being more or equally certain to take great values (in the previous sense) than a loss $Y \in \chi$, the "risk"[2] $\rho(X)$ associated to the loss X by a risk measure $\rho : \chi \longrightarrow \mathbb{R}$ having the property of *consistency with respect to the* $\leq_{\mathrm{mon},\mu}$ *−relation* is greater than or equal to the "risk" $\rho(Y)$ associated to the loss Y.
Thanks to proposition 3.2.1, an analogous interpretation could be given of the $\leq_{sl,\mu}$ −relation and of the property of consistency with respect to the $\leq_{sl,\mu}$ −relation.

3.3.3 Characterizing risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation

We have seen that, for a given capacity μ, the set of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation is included in the set of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{mon,\mu}$ −relation. Besides, in the case where the initial capacity μ is continuous from below and from above, a characterization of the latter set in terms of Choquet integrals with respect to a distorted capacity has been established in theorem 3.3.1 combined with remark 3.3.8. This subsection is devoted to a characterization of the former set of risk measures in terms of Choquet integrals with respect to a distorted capacity where the distortion function is concave. Two separate theorems, corresponding to the two implications of which the characterization consists, are presented.
The following theorem is a representation result for monetary risk measures satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation.

2. The expression *"the risk" of a loss* $X \in \chi$ designates here the number $\rho(X)$ associated to X by a risk measure ρ.

Theorem 3.3.2 *Let μ be a capacity. Assume that there exists a real-valued measurable function Z such that the distribution function G_Z of Z is continuous and satisfies the following property :* $\lim_{x\to-\infty} G_Z(x) = 0$ *and* $\lim_{x\to+\infty} G_Z(x) = 1$.
If $\rho : \chi \to \mathbb{R}$ is a monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$ −relation, then there exists a concave *distortion function ψ such that*

$$\rho(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi.$$

The proof of this theorem is based on the representation result of lemma 3.3.1, on proposition 3.2.1, and on lemma 3.3.2 below. The lemma 3.3.2 is well-known in the classical case of a probability measure as a way of constructing a random variable with a uniform distribution on $[0, 1]$.

Lemma 3.3.2 *Let μ be a capacity. Assume that there exists a real-valued measurable function Z such that the distribution function G_Z of Z is continuous and satisfies*

$$\lim_{x\to-\infty} G_Z(x) = 0 \ \ \textit{and} \ \ \lim_{x\to+\infty} G_Z(x) = 1. \tag{3.3.3}$$

Set $U := G_Z(Z)$. The distribution function G_U of U is given by : $G_U(x) = \begin{cases} 0, & \textit{if } x < 0 \\ x, & \textit{if } x \in [0,1] \\ 1, & \textit{if } x > 1. \end{cases}$

Proof of lemma 3.3.2 : The measurable function U can be written in the following manner : $U = f(Z)$, where, for the easing of the presentation, we have set $f := G_Z$. As in the proof of lemma 1.3.1, we define the upper generalized inverse $\check{f}$ of the non-decreasing function f by $\check{f}(x) := \inf\{y \in \mathbb{R} : f(y) > x\},\ \forall x \in \mathbb{R}$. The function f being non-decreasing and continuous, we know from the proof of proposition 3.2 in Yan (2009) that for all $x \in \mathbb{R}$, $G_U(x) = G_{f(Z)}(x) = G_Z \circ \check{f}(x)$.
Therefore, for all $x \in \mathbb{R}$, $G_U(x) = G_Z \circ \check{G}_Z(x)$.
Now, according to the definitions of $\check{G}_Z$ and of the upper quantile function r_Z^+, we have $\check{G}_Z(x) = r_Z^+(x), \forall x \in (0, 1)$. Moreover, thanks to the assumption (3.3.3), $r_Z^+(x)$ belongs to $\mathbb{R}$, $\forall x \in (0, 1)$.
Thus, if $x \in (0, 1)$, then $G_Z \circ \check{G}_Z(x) = G_Z \circ r_Z^+(x) = x$. The last equality in the previous computation is due to the continuity of G_Z on $\mathbb{R}$.
If $x \geq 1$, then $\check{G}_Z(x) = +\infty$ and $G_Z \circ \check{G}_Z(x) = 1$.
If $x < 0$, then $\check{G}_Z(x) = -\infty$ and $G_Z \circ \check{G}_Z(x) = G_Z(-\infty) = 1 - \mu(Z > -\infty) = 0$.
Finally, if $x = 0$, then either $\check{G}_Z(0) = -\infty$ or $\check{G}_Z(0) \in \mathbb{R}$. In both of the situations, $G_Z \circ \check{G}_Z(0) = 0$.
The expression for G_U is thus proved. $\square$

The following two remarks concern the assumptions of the previous lemma.

Remark 3.3.11 The existence of a measurable function Z on $(\Omega, \mathcal{F})$ with a continuous distribution function with respect to the capacity μ has been assumed in the previous lemma 3.3.2. In the "classical" case where μ is a probability measure this assumption is equivalent to the usual assumption of non-atomicity of the measure space $(\Omega, \mathcal{F}, \mu)$ (cf. Föllmer and Schied 2004).

Remark 3.3.12 We note that assumption (3.3.3) of the previous lemma is not redundant in the case of a capacity μ which is not a probability measure. We also note that if μ and Z do not satisfy the assumption (3.3.3), the result on the distribution function G_U of U of the lemma may not hold true. Indeed, let us consider the following counter-example. Let $(\Omega, \mathcal{F}, P)$ be a probability space such that there exists a random variable Z whose distribution function F_Z (with respect to P) is continuous and satisfies $0 < F_Z(x) < 1, \forall x \in \mathbb{R}$. Let μ be a capacity of the form $\mu := \psi \circ P$ where ψ is a distortion function which is continuous on $(0,1)$ and such that $b := \sup_{x<1} \psi(x) < 1$. Then, the distribution function $G_{Z,\mu}$ of Z (with respect to μ) is continuous but fails to satisfy the assumption (3.3.3) in lemma 3.3.2 as

$$\begin{aligned}\lim_{x \to -\infty} G_{Z,\mu}(x) &= \lim_{x \to -\infty} \Big(1 - \psi(1 - F_Z(x))\Big) = 1 - \sup_{x \in \mathbb{R}} \psi(1 - F_Z(x)) \\ &= 1 - \sup_{y<1} \psi(y) = 1 - b > 0.\end{aligned}$$

Let us compute $G_{U,\mu}(x)$ for $x \in (0, 1-b)$. For $x \in (0, 1-b)$, $\check{G}_{Z,\mu}(x) = r^{+}_{Z,\mu}(x) = -\infty$. Therefore, for $x \in (0, 1-b)$, $G_{U,\mu}(x) = G_{Z,\mu} \circ \check{G}_{Z,\mu}(x) = G_{Z,\mu}(-\infty) = 0 \neq x$ which provides the desired counter-example.

Let us now prove theorem 3.3.2.

Proof of theorem 3.3.2 : It is easy to check that the monetary risk measure ρ satisfies the properties (i), (ii) and (iii) in lemma 3.3.1. Therefore, there exists a distortion function ψ such that $\rho(X) = \mathbb{E}_{\psi \circ \mu}(X)$, $\forall X \in \chi$. It remains to show that the distortion function ψ is concave.

Let $x \in [0,1]$ and $y \in [0,1]$ be such that $x < y$. There exist measurable sets A and B satisfying the following properties : $A \subset B$, $\mu(A) = x$ and $\mu(B) = y$. Indeed, if we set $A := \{U > 1 - x\}$ and $B := \{U > 1 - y\}$ where $U := G_Z(Z)$, we have that $A \subset B$. Moreover, according to lemma 3.3.2, $\mu(A) = \mu(U > 1-x) = 1-G_U(1-x) = 1-(1-x) = x$. Similarly, we compute $\mu(B) = y$. Therefore, the sets A and B are as desired.

Furthermore, there exists a measurable set C such that $\mu(C) = \frac{x+y}{2}$ (the set C can be constructed by setting $C := \{U > 1 - \frac{x+y}{2}\}$).

We now set $X := \frac{1}{2}\mathbb{I}_A + \frac{1}{2}\mathbb{I}_B$ and $Y := \mathbb{I}_C$ and we note that the measurable functions $\frac{1}{2}\mathbb{I}_A$

and $\frac{1}{2}\mathbb{I}_B$ are comonotonic as $A \subset B$.
Let us show that $X \leq_{sl,\mu} Y$. According to proposition 3.2.1, it suffices to prove that $\forall t \in (0,1)$,

$$\int_t^1 r_X^+(s)ds \leq \int_t^1 r_Y^+(s)ds. \qquad (3.3.4)$$

Now, $r_Y^+(t) = \mathbb{I}_{[1-\mu(C),1)}(t)$ and $r_X^+(t) = \frac{1}{2}\mathbb{I}_{[1-\mu(A),1)}(t) + \frac{1}{2}\mathbb{I}_{[1-\mu(B),1)}(t)$ for almost every t where lemma 1.2.2 and lemma 1.3.1 have been used to compute $r_X^+(t)$. Therefore, equation (3.3.4) is equivalent to $\frac{1}{2}\Big(1-\max\{t,1-\mu(A)\}\Big)+\frac{1}{2}\Big(1-\max\{t,1-\mu(B)\}\Big) \leq 1-\max\{t,1-\mu(C)\}$ which is equivalent to $\frac{1}{2}\min\{1-t,\mu(A)\} + \frac{1}{2}\min\{1-t,\mu(B)\} \leq \min\{1-t,\mu(C)\}$. The observation that, for a fixed $t \in (0,1)$, the mapping $z \longrightarrow \min\{1-t,z\}$ is concave allows us to conclude that equation (3.3.4) holds true.
The consistency of ρ with respect to the $\leq_{sl,\mu}$ $-$relation implies that $\rho(X) \leq \rho(Y)$ which is equivalent to $\mathbb{E}_{\psi\circ\mu}(\frac{1}{2}\mathbb{I}_A+\frac{1}{2}\mathbb{I}_B) \leq \mathbb{E}_{\psi\circ\mu}(\mathbb{I}_C)$. The positive homogeneity and the comonotonic additivity of the Choquet integral then give $\frac{1}{2}\psi\circ\mu(A)+\frac{1}{2}\psi\circ\mu(B) \leq \psi\circ\mu(C)$. The concavity of ψ follows as $\mu(A) = x$, $\mu(B) = y$, $\mu(C) = \frac{x+y}{2}$ and as x and y are arbitrary.

□

Remark 3.3.13 The distortion function in the representation result of the previous theorem (theorem 3.3.2) is unique. Indeed, suppose that there exists a distortion function $\tilde{\psi}$ such that $\rho(X) = \mathbb{E}_{\tilde{\psi}\circ\mu}(X), \forall X \in \chi$. Let $x \in [0,1]$. Under the assumptions of theorem 3.3.2 there exists a measurable set A such that $\mu(A) = x$ (see the proof of theorem 3.3.2 for the construction of the set A). On the other hand, $\rho(\mathbb{I}_A) = \psi\circ\mu(A) = \tilde{\psi}\circ\mu(A)$, which implies the desired equality, namely $\psi(x) = \tilde{\psi}(x)$.

Remark 3.3.14 One may wonder if the Choquet integral with respect to a distorted capacity of the form $\psi \circ \mu$ (as the one which appears in the representation formula of theorem 3.3.2) can be compared with the Choquet integral with respect to the initial capacity μ. In the case where the distortion function ψ is concave (which is the case in the representation formula of theorem 3.3.2), the following inequality holds : $\psi \circ \mu(A) \geq \mu(A), \forall A \in \mathcal{F}$. Therefore, $\mathbb{E}_{\psi\circ\mu}(X) \geq \mathbb{E}_\mu(X), \forall X \in \chi$. We conclude that, under the assumptions of theorem 3.3.2, a monetary risk measure ρ having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ $-$relation satisfies the property : $\rho(X) \geq \mathbb{E}_\mu(X), \forall X \in \chi$.

In the particular case where, along with the assumptions made in theorem 3.3.2, the additional assumption of concavity of the capacity μ is made, a monetary risk measure ρ satisfying the properties of theorem 3.3.2, namely comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ $-$relation, is necessarily a convex monetary risk measure. The result is formulated in the following corollary. The convexity of ρ in this case is due to the concavity of the distorted capacity $\psi \circ \mu$ in the representation of ρ and to the sub-additivity of the Choquet integral with respect to a concave capacity. For the corresponding result in the "classical" case of a probability the reader is referred to Song and Yan (2009 a.), as well as to Föllmer and Schied (2004).

Corollary 3.3.1 *Let μ be a concave capacity and assume that there exists a real-valued measurable function Z such that the distribution function G_Z of Z is continuous and satisfies the following property :* $\lim_{x\to-\infty} G_Z(x) = 0$ *and* $\lim_{x\to+\infty} G_Z(x) = 1$.
Let $\rho : \chi \to \mathbb{R}$ be a monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$ −relation. Then ρ is a convex *monetary risk measure on χ.*

Remark 3.3.15 We note that if, along with the assumptions on the space $(\Omega, \mathcal{F}, \mu)$ in the previous theorem 3.3.2 (respectively in corollary 3.3.1), the additional assumption of continuity from below and from above on the capacity μ is made, then the property of consistency with respect to the $\leq_{sl,\mu}$ −relation in theorem 3.3.2 (resp. corollary 3.3.1) can be replaced by the property of consistency with respect to the $\leq_{icx,\mu}$ −relation. The statement is due to the second assertion in proposition 3.3.2. We note, furthermore, that the assumption on the limits of the distribution function G_Z of Z in theorem 3.3.2 (resp. corollary 3.3.1) is made redundant by this additional continuity assumption on the capacity μ (cf. remark 2.2.1).

It has been established in the previous theorem 3.3.2 that, under suitable assumptions on the initial space $(\Omega, \mathcal{F}, \mu)$, a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation can be represented as a Choquet integral with respect to a distorted capacity of the form $\psi \circ \mu$ where the distortion function ψ is concave. In order to complete the desired characterization it remains to show that the converse statement holds true which is the purpose of the following theorem.

Theorem 3.3.3 *Let μ be a capacity and let ψ be a concave distortion function. The functional ρ defined by $\rho(X) := \mathbb{E}_{\psi\circ\mu}(X), \forall X \in \chi$ is a monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$ −relation.*

The following lemma will be used in the proof of theorem 3.3.3. The lemma is a generalization of a well-known "classical" expression for Choquet integrals with respect to a distorted probability whose distortion function is concave (see, for instance, Föllmer and Schied 2004 or Carlier and Dana 2006 for the classical case). Our proof follows the proof given by Föllmer and Schied (2004) and is included for reader's convenience.

Lemma 3.3.3 *Let μ be a capacity and let ψ be a concave distortion function. For all $X \in \chi$,*

$$\mathbb{E}_{\psi\circ\mu}(X) = \psi(0+) \sup_{t<1} r_X^+(t) + \int_0^1 \psi'(1-t) r_X^+(t)dt \tag{3.3.5}$$

Proof of the lemma : It suffices to prove equation (3.3.5) for non-negative elements of χ, the terms on both sides of the equality being translation invariant. Let X be in χ_+. The following expression is similar to the "classical" one ; the proof is due to the non-decreasingness of G_X and to the definition of r_X^+ and is left to the reader :

$$r_X^+(t) = \int_0^{+\infty} \mathbb{I}_{\{G_X(s)\leq t\}} ds, \ \forall t \in (0,1). \tag{3.3.6}$$

Thanks to (3.3.6) we compute

$$\begin{aligned}\int_0^1 \psi'(1-t) r_X^+(t)dt &= \int_0^1 \psi'(1-t) \int_0^{+\infty} \mathbb{I}_{\{G_X(s)\leq t\}} ds\ dt \\ &= \int_0^{+\infty} \int_0^{1-G_X(s)} \psi'(y)dy\ ds \\ &= \int_0^{+\infty} \Big(\psi(1-G_X(s)) - \psi(0+)\Big) \mathbb{I}_{\{G_X(s)<1\}} ds\end{aligned}$$

where the equation $\int_0^y \psi'(s)ds = (\psi(y) - \psi(0+))\mathbb{I}_{y>0}$ has been used to obtain the last line. Using the definition of the Choquet integral and the fact that

$$\sup_{t<1} r_X^+(t) = \int_0^{+\infty} \mathbb{I}_{\{G_X(s)<1\}} ds$$

whose proof is left to the reader, we obtain

$$\begin{aligned}\int_0^1 \psi'(1-t) r_X^+(t)dt &= \int_0^{+\infty} \psi(1-G_X(s))ds - \psi(0+) \int_0^{+\infty} \mathbb{I}_{\{G_X(s)<1\}} ds \\ &= \mathbb{E}_{\psi\circ\mu}(X) - \psi(0+) \sup_{t<1} r_X^+(t).\end{aligned}$$

The lemma is thus proved.

□

Proof of theorem 3.3.3 : As recalled in proposition 1.2.2, the Choquet integral satisfies the properties of monotonicity, translation invariance and comonotonic additivity. Therefore, the only property of the functional ρ which has to be proved is the property of consistency with respect to the $\leq_{sl,\mu}$ −relation.
Let $X, Y \in \chi$ be such that $X \leq_{sl,\mu} Y$. Let us prove that $\mathbb{E}_{\psi\circ\mu}(X) \leq \mathbb{E}_{\psi\circ\mu}(Y)$ which, thanks to lemma 3.3.3, is equivalent to

$$\psi(0+) \sup_{t<1} r_X^+(t) + \int_0^1 \psi'(1-t) r_X^+(t)dt \leq \psi(0+) \sup_{t<1} r_Y^+(t) + \int_0^1 \psi'(1-t) r_Y^+(t)dt.$$

Proposition 2.3.5 implies that $\int_0^1 \psi'(1-t) r_X^+(t)dt \leq \int_0^1 \psi'(1-t) r_Y^+(t)dt$. The number $\psi(0+)$ being non-negative, it remains to show that $\sup_{t<1} r_X^+(t) \leq \sup_{t<1} r_Y^+(t)$.

Suppose, by way of contradiction, that $\sup_{t<1} r_X^+(t) > \sup_{t<1} r_Y^+(t)$. Then, there exists $t_0 \in [0,1)$ such that $r_X^+(s) \geq r_X^+(t_0) > \sup_{t<1} r_Y^+(t)$, $\forall s \geq t_0$. This implies that $r_X^+(s) > r_Y^+(s)$, $\forall s \geq t_0$ leading to $\int_{t_0}^1 (r_X^+(s) - r_Y^+(s))ds > 0$. The last inequality contradicts the relation $X \leq_{sl,\mu} Y$ (cf. the characterization of the $\leq_{sl,\mu}$ −relation in proposition 3.2.1). The previous reasoning leads to the desired implication, namely $X \leq_{sl,\mu} Y \Rightarrow \sup_{t<1} r_X^+(t) \leq \sup_{t<1} r_Y^+(t)$, and concludes the proof.

$\square$

3.3.4 Convex generalized distortion risk measures : a counter-example

As recalled in remark 3.3.5, a generalized distortion risk measure of the form $\mathbb{E}_{\psi\circ\mu}(\cdot)$ is convex if and only if the distorted capacity $\psi \circ \mu$ is concave in the sense of definition 1.2.2. The purpose of this section is to investigate the question whether the concavity of a distorted capacity $\psi \circ \mu$ (and therefore, the convexity of $\mathbb{E}_{\psi\circ\mu}(\cdot)$) can be characterized by means of the concavity of the distortion function ψ.

It has been seen in example 2.2.1 of the previous chapter 2 that, in the case where μ is a concave capacity, a distorted capacity of the form $\psi\circ\mu$ is concave if the distortion function ψ is concave. On the other hand, it is well-known that in the "classical" case where μ is a probability measure, under a non-atomicity assumption on the measure space $(\Omega, \mathcal{F}, \mu)$, the converse statement also holds true, namely the concavity of a distorted probability of the form $\psi \circ \mu$ implies the concavity of the distortion function ψ (cf. proposition 4.69 in Föllmer and Schied 2004).

Nevertheless, in the more general case where μ is a concave capacity which is not necessarily a probability measure, this converse statement may not be true even if the existence of a measurable function Z with a continuous distribution function $G_Z := G_{Z,\mu}$ is assumed. Let us consider the following counter-example.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space. Let ϕ be a distortion function which is concave and continuous and set $\mu := \phi \circ \mathbb{P}$. Then, the capacity μ is a concave capacity, the distortion function ϕ being concave. Furthermore, μ is continuous from below and from above, the function ϕ being continuous. Moreover, there exists a measurable function Z on $(\Omega, \mathcal{F})$ such that the distribution function (with respect to μ) $G_Z := G_{Z,\mu}$ of Z is continuous (in fact, one can easily verify that any random variable Z whose distribution function with respect to $\mathbb{P}$ is continuous satisfies this property ; the existence of such a random variable is guaranteed by the non-atomicity assumption on $(\Omega, \mathcal{F}, \mathbb{P})$).

To be more concrete, let us specify the definition of ϕ : $\phi(x) := x^\beta, \forall x \in [0,1]$ where $\beta \in (0,1)$. Let us further define a distortion function $\psi : [0,1] \to [0,1]$ by $\psi(x) := x^{\frac{\alpha}{\beta}}, \forall x \in [0,1]$ where $\alpha \in (0,1)$ is such that $\alpha > \beta$. Let us consider the distorted capacity $\psi \circ \mu$ where $\mu := \phi \circ \mathbb{P}$ as above.

The distortion function ψ is not concave ; in fact, ψ is a strictly convex function. Nevertheless, the distorted capacity $\psi \circ \mu$ is a concave capacity. The latter property is easily obtained by observing that $\psi \circ \mu = (\psi \circ \phi) \circ \mathbb{P}$ and that $\psi \circ \phi$ is a concave distortion function as $\psi \circ \phi(x) = x^\alpha, \forall x \in [0,1]$. Thus, the capacity $\psi \circ \mu$ is concave as it can be represented as a distorted probability with respect to a concave distortion function.
To summarize, we have given an example of a measurable space $(\Omega, \mathcal{F})$ endowed with a capacity μ which is concave, continuous from below and from above (but not necessarily additive) and such that there exists a measurable function whose distribution function with respect to μ is continuous. We have then shown that it is possible to construct a distorted capacity of the form $\psi \circ \mu$ which is concave (in the sense of definition 1.2.2) but whose distortion function ψ is not concave thus providing the desired counter-example.

3.4 "Kusuoka-type" characterization of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$-relation

The purpose of this section is to provide a "Kusuoka-type" characterization of the class of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$-relation under suitable assumptions on the space $(\Omega, \mathcal{F}, \mu)$ where μ is a capacity. We recall, for reader's convenience, the classical Kusuoka's result (cf. theorem 7 in Kusuoka 2001) in a form which is given in Ekeland and Schachermayer (2011) (theorem 1.4) :

Theorem 3.4.1 (Kusuoka's theorem) *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ be a given functional. Then, the following two statements are equivalent :*

(i) *The functional ρ is a convex monetary risk measure having the properties of comonotonic additivity and law-invariance.*

(ii) *There exists $\alpha \in [0,1]$ and a random variable $Y \in L^1_+(\Omega, \mathcal{F}, P)$ satisfying $\mathbb{E}_P(Y) = 1$ such that*

$$\rho(X) = \alpha \; ess \; sup(X) + (1-\alpha)\, \rho_Y(X), \quad \forall X \in L^\infty(\Omega, \mathcal{F}, P),$$

where $\rho_Y(X) := \sup_{\tilde{X} \in L^\infty(\Omega,\mathcal{F},P):\tilde{X}\sim X} \mathbb{E}_P(Y\tilde{X})$ and the notation $\tilde{X} \sim X$ means that $\tilde{X}$ and X have the same law (with respect to P).

Let us further remark that the law-invariance property in statement (i) of the previous theorem can be replaced by the property of consistency with respect to the "classical" stop-loss order relation $\leq_{sl,\ P}$ (with respect to the probability P). More precisely, in the case where the probability space $(\Omega, \mathcal{F}, P)$ is atomless, the following well-known result holds true ; the result is recalled for reader's convenience.

Proposition 3.4.1 *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ be a given functional. Then, the following statements are equivalent :*

(i) *The functional ρ is a convex monetary risk measure having the properties of comonotonic additivity and law-invariance.*

(ii) *The functional ρ is a convex monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\text{mon, P}}$-relation.*

(iii) *The functional ρ is a convex monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\text{sl, P}}$-relation.*

(iv) *The functional ρ is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\text{sl, P}}$-relation.*

Proof : The equivalence between assertions (iii) and (iv) is a consequence of corollary 3.3.1 applied to the particular case of an atomless probability space. The implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are obvious. The implication $(i) \Rightarrow (iii)$ can be found in Cherny and Grigoriev (2007) (page 294).

□

Thus, theorem 3.4.1 can be viewed as a way of characterizing (convex) monetary risk measures having the properties of comonotonic additivity and consistency with respect to the "classical" $\leq_{sl,\ P} -$ relation in the case where the probability space $(\Omega, \mathcal{F}, P)$ is atomless.
We note as well that, thanks to lemma 4.5.5. in Föllmer and Schied (2004), statement (ii) in theorem 3.4.1 can be reformulated in the following manner :

(ii bis) There exists $\alpha \in [0, 1]$ and a random variable $Y \in L^1_+(\Omega, \mathcal{F}, P)$ satisfying $\mathbb{E}_P(Y) = 1$ such that

$$\rho(X) = \alpha \text{ ess sup}(X) + (1 - \alpha) \int_0^1 q_Y(t) q_X(t) dt, \quad \forall X \in L^\infty(\Omega, \mathcal{F}, P),$$

where q_X (resp. q_Y) denotes (the) quantile function of X (resp. Y) with respect to the probability P.

Thanks to the previous considerations, theorem 3.4.1 can be reformulated as follows :

Theorem 3.4.2 (Kusuoka's theorem - equivalent formulation) *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ be a given functional. Then the following two statements are equivalent :*

(i) *The functional ρ is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\text{sl, P}}$-relation.*

(ii) *There exists $\alpha \in [0,1]$ and a random variable $Y \in L^1_+(\Omega, \mathcal{F}, P)$ satisfying $\mathbb{E}_P(Y) = 1$ such that*

$$\rho(X) = \alpha \; ess\ sup(X) + (1-\alpha) \int_0^1 q_Y(t) q_X(t) dt, \quad \forall X \in L^\infty(\Omega, \mathcal{F}, P),$$

where q_X (resp. q_Y) denotes (the) quantile function of X (resp. Y) with respect to P.

A "generalization" of theorem 3.4.2 to the setting of a capacity (which is not necessarily a probability measure) is established in the following theorem.

Theorem 3.4.3 (Kusuoka-type characterization in the case of a capacity) *Let μ be a capacity. Assume that there exists a real-valued measurable function Z such that the distribution function G_Z of Z is continuous and satisfies the following property : $\lim_{x \to -\infty} G_Z(x) = 0$ and $\lim_{x \to +\infty} G_Z(x) = 1$.*
Let $\rho : \chi \longrightarrow \mathbb{R}$ be a given functional. Then the following two statements are equivalent :

(i) *The functional ρ is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\text{sl},\mu}$-relation.*

(ii) *There exists $\alpha \in [0,1]$ and a non-negative measurable function Y satisfying $\int_0^1 r_{Y,\mu}(t)dt = 1$ such that*

$$\rho(X) = \alpha \sup_{t<1} r^+_{X,\mu}(t) + (1-\alpha) \int_0^1 r_{Y,\mu}(t) r_{X,\mu}(t) dt, \quad \forall X \in \chi.$$

The following lemma summarizes some of the main properties of the functional $X \mapsto \sup_{t<1} r^+_{X,\mu}(t)$ and will be used in the proof of theorem 3.4.3.

Lemma 3.4.1 *Let μ be a capacity. The functional $\rho_\infty : \chi \longrightarrow \mathbb{R}$ defined by $\rho_\infty(X) := \sup_{t<1} r^+_X(t), \forall X \in \chi$ is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\text{sl},\mu}$ $-$relation.*
Moreover, the functional ρ_∞ can be represented in the following manner :

$$\rho_\infty(X) = \mathbb{E}_{\psi \circ \mu}(X), \forall X \in \chi$$

where ψ is a concave distortion function given by $\psi(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0. \end{cases}$

Proof of the lemma : The translation invariance of the functional ρ_∞ follows from lemma 1.3.1. The monotonicity of ρ_∞ is due to the definition of the upper quantile function and to the monotonicity of the capacity μ.

Let us prove the comonotonic additivity of ρ_∞. Let X and Y be two comonotonic functions in χ. According to proposition 1.2.1, there exists $Z \in \chi$ and two non-decreasing continuous functions f and g on $\mathbb{R}$ such that $X = f(Z)$ and $Y = g(Z)$. Therefore,

$$\rho_\infty(X+Y) = \sup_{t<1} r^+_{X+Y}(t) = \sup_{t<1} r^+_{(f+g)(Z)}(t) = \sup_{t<1}(f+g)(r^+_Z(t))$$

where lemma 1.3.1 has been used to obtain the last equality.
As the function $f+g$ is non-decreasing and continuous on $\mathbb{R}$ and as $\sup_{t<1} r^+_Z(t) \in \mathbb{R}$, we have $\sup_{t<1}(f+g)(r^+_Z(t)) = (f+g)(\sup_{t<1} r^+_Z(t))$. The same argument is used to show that $f\Big(\sup_{t<1} r^+_Z(t)\Big) = \sup_{t<1} f(r^+_Z(t))$ and $g\Big(\sup_{t<1} r^+_Z(t)\Big) = \sup_{t<1} g(r^+_Z(t))$. Thus,

$$\begin{aligned}\sup_{t<1}(f+g)(r^+_Z(t)) &= (f+g)(\sup_{t<1} r^+_Z(t)) = \sup_{t<1} f(r^+_Z(t)) + \sup_{t<1} g(r^+_Z(t)) = \\ &= \sup_{t<1} r^+_{f(Z)}(t) + \sup_{t<1} r^+_{g(Z)}(t) = \sup_{t<1} r^+_X(t) + \sup_{t<1} r^+_Y(t)\end{aligned}$$

where lemma 1.3.1 has been used again to obtain the last but one equality. The comonotonic additivity of ρ_∞ is thus proved.
The property of consistency with respect to the $\leq_{sl,\mu}-$ relation has already been shown at the end of the proof of theorem 3.3.3.
Finally, an application of Schmeidler's representation theorem (theorem 3.2.1) gives the existence of a capacity ν such that $\rho_\infty(X) = \mathbb{E}_\nu(X)$, $\forall X \in \chi$. The capacity ν is given by

$$\nu(A) = \rho_\infty(\mathbb{I}_A) = \sup_{t<1} r^+_{\mathbb{I}_A}(t) = \sup_{t<1} \mathbb{I}_{[1-\mu(A),1)}(t) = \begin{cases} 1, \text{ if } \mu(A) > 0 \\ 0, \text{ if } \mu(A) = 0. \end{cases}$$

Thus, $\nu(A) = \psi(\mu(A))$ which concludes the proof.

□

Some of the main properties of the functional $X \longmapsto \int_0^1 r_Y(t) r_X(t)dt$ (for a given $Y \geq 0$ such that $\int_0^1 r_Y(t)dt = 1$) have already been studied in subsection 2.4.1 of the previous chapter (cf. also subsection 5.1 of Grigorova 2010) and are summarized in the following lemma for reader's convenience.

Lemma 3.4.2 *Let μ be a capacity. Let Y be a non-negative measurable function such that $\int_0^1 r_Y(t)dt = 1$. The functional $\rho^Y : \chi \longrightarrow \mathbb{R}$ defined by $\rho^Y(X) := \int_0^1 r_Y(t) r_X(t)dt, \forall X \in \chi$ is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$ $-$relation.*
Moreover, the functional ρ^Y can be represented in the following manner :

$$\rho^Y(X) = \mathbb{E}_{\psi^Y \circ \mu}(X), \forall X \in \chi$$

where ψ^Y is a concave distortion function given by $\psi^Y(x) = \int_{1-x}^1 r_Y(t)dt, \forall x \in [0,1]$.

Let us now prove theorem 3.4.3.

Proof of theorem 3.4.3 : The implication $(ii) \Rightarrow (i)$ is a consequence of lemma 3.4.1 and lemma 3.4.2.

To prove the converse implication, let ρ be a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation. Thanks to theorem 3.3.2 and to lemma 3.3.3, there exists a concave distortion function ψ such that $\forall X \in \chi$,

$$\rho(X) = \psi(0+) \sup_{t<1} r_X^+(t) + \int_0^1 \psi'(1-t) r_X^+(t) dt.$$

– If $\psi(0+) = 1$, then $\rho(X) = \sup_{t<1} r_X^+(t)$, $\forall X \in \chi$ which proves the desired result with $\alpha = 1$.
– Otherwise, by setting $\alpha := \psi(0+)$, we have

$$\rho(X) = \alpha \sup_{t<1} r_X^+(t) + (1-\alpha) \int_0^1 \frac{\psi'(1-t)}{1-\psi(0+)} r_X^+(t) dt, \ \forall X \in \chi.$$

Let us remark that $\int_0^1 \frac{\psi'(1-t)}{1-\psi(0+)} dt = 1$. Therefore, in order to prove statement (ii), it suffices to prove that there exists a non-negative measurable function Y such that $r_Y(t) = \frac{\psi'(1-t)}{1-\psi(0+)}$ for almost every $t \in (0,1)$.

Set $U := G_Z(Z)$ and define a function g by setting $g(t) := \frac{\psi'_+(1-t)}{1-\psi(0+)}$, $\forall t \in (0,1)$ where ψ'_+ denotes the right-hand derivative of the concave function ψ. Let Y be defined by $Y := g(U)$ (where, in order to assure that Y is well-defined on Ω, the definition of g has been extended to $[0,1]$ by setting $g(0) := \lim_{t\downarrow 0} \frac{\psi'_+(1-t)}{1-\psi(0+)}$ and $g(1) := \lim_{t\uparrow 1} \frac{\psi'_+(1-t)}{1-\psi(0+)}$).

Then, the measurable function Y is as wanted. Indeed, $Y \geq 0$. Moreover, the distribution function G_U of U being continuous (according to lemma 3.3.2) and the function g being non-decreasing, we can apply lemma 1.2.1 to obtain :

$$r_Y(t) = r_{g(U)}(t) = g(r_U(t)) \text{ for almost every } t \in (0,1). \tag{3.4.1}$$

Now, it can be deduced from lemma 3.3.2 that $r_U(t) = t$ for all $t \in (0,1)$. This observation combined with equality (3.4.1) allows to conclude that $r_Y(t) = g(t)$ for almost every $t \in (0,1)$.

□

Remark 3.4.1 Let us remark that, unlike the classical case of proposition 3.4.1, under the more general assumptions on $(\Omega, \mathcal{F}, \mu)$ of theorem 3.4.3 a monetary risk measure ρ satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation (as the one of statement (i) in theorem 3.4.3) is not necessarily convex. A counter-example similar to the one constructed in subsection 3.3.4 is given in the appendix.

Let us recall, nevertheless, that if, along with the assumptions made in theorem 3.4.3, the assumption of concavity of the capacity μ is made, a monetary risk measure ρ satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation is convex (cf. corollary 3.3.1).

Remark 3.4.2 Let us remark also that, unlike the classical case of proposition 3.4.1, under the more general assumptions on $(\Omega, \mathcal{F}, \mu)$ of theorem 3.4.3 a convex monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{mon,\mu}$ −relation is not necessarily consistent with respect to the $\leq_{sl,\mu}$ −relation even if the additional assumption of concavity of the capacity μ is made. A counter-example, based on the one of subsection 3.3.4, is given in the appendix.

One may wonder if, in our setting of a capacity (which is not necessarily a probability measure), statement (ii) in theorem 3.4.3 could be linked to the value function of an optimization problem analogous to the one appearing in statement (ii) of the "classical" Kusuoka's theorem (theorem 3.4.1). The following result has been established in chapter 2 (cf. also Grigorova 2010) : the formulation given hereafter is suitable for the needs of the present chapter and is due to theorem 2.4.1 combined with remark 2.4.1, remark 2.4.3 and proposition 2.3.2 of chapter 2.

Proposition 3.4.2 *Let μ be a capacity which is assumed to be concave and continuous from below and from above. Let Y be a given non-negative measurable function such that $\int_0^1 r_{Y,\mu}(t)dt = 1$. Then the funcional $\rho_Y : \chi_+ \longrightarrow \mathbb{R}$ defined by*

$$\rho_Y(X) := \sup_{\tilde{X} \in \chi_+ : \tilde{X} \leq_{\mathrm{sl},\mu} X} \mathbb{E}_\mu(Y\tilde{X}), \ \forall X \in \chi_+$$

can be expressed in the following manner : $\rho_Y(X) = \int_0^1 r_{Y,\mu}(t) r_{X,\mu}(t)dt$.

The previous proposition 3.4.2 combined with theorem 3.4.3 and remark 2.2.1 leads to the following

Theorem 3.4.4 *Let μ be a capacity which is assumed to be* concave *and continuous from below and from above and assume that there exists a real-valued measurable function Z on $(\Omega, \mathcal{F})$ such that the distribution function G_Z of Z (with respect to μ) is continuous. Let $\rho : \chi_+ \longrightarrow \mathbb{R}$ be a given functional. Then the following two statements are equivalent :*

(i) *The functional ρ is a (convex) monetary risk measure on χ_+ having the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$-relation.*

(ii) *There exists $\alpha \in [0,1]$ and a non-negative measurable function Y satisfying $\int_0^1 r_{Y,\mu}(t)dt = 1$ such that*

$$\rho(X) = \alpha \sup_{t<1} r^+_{X,\mu}(t) + (1-\alpha)\,\rho_Y(X), \quad \forall X \in \chi_+,$$

where $\rho_Y(X) := \sup_{\tilde{X}\in\chi_+:\tilde{X}\leq_{\mathrm{sl},\mu}X} \mathbb{E}_\mu(Y\tilde{X}),\ \forall X \in \chi_+.$

The previous theorem may be seen as an analogue of theorem 3.4.1 in the setting of a capacity which is assumed to be concave and continuous from below and from above.

3.5 Some examples of generalized distortion risk measures

In this section some generalizations to the case of a capacity of some well-known "classical" risk measures are given.

3.5.1 A "generalized" Value at Risk

Let us recall, for reader's convenience, the well-known "classical" definition of the Value at Risk at level $\lambda \in (0,1)$ with respect to a given probability P of a given "potential loss" $X \in \chi$ (denoted by $VaR_\lambda(X)$ or $VaR_\lambda^P(X)$) :

$$VaR_\lambda(X) := q_X^-(\lambda),$$

where, as before, the symbol q_X^- stands for the lower quantile function of X with respect to the probability P. The same sign convention in the definition of the $VaR_\lambda(X)$ as the one used in the present chapter is used, for instance, by Dhaene et al. (2006) or Song and Yan (2009 a.).
We now consider a generalization of the previous definition to the case of a capacity which is not necessarily a probability measure. The definition and some properties of the "generalized" Value at Risk are given in the following

Definition/Proposition 3.5.1 *Let* μ *be a capacity on* $(\Omega,\mathcal{F})$ *and* λ *be in* $(0,1)$. *The functional* $GVaR_\lambda^\mu : \chi \to \mathbb{R}$ *defined by*

$$GVaR_\lambda^\mu(X) := r_{X,\mu}^-(\lambda),\ \forall X \in \chi$$

is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{mon},\mu}$ *−relation. Moreover, the functional* $GVaR_\lambda^\mu$ *has the following representation*

$$GVaR_\lambda^\mu(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi \tag{3.5.1}$$

where $\psi(x) := \psi_\lambda(x) := \mathbb{I}_{(1-\lambda,1]}(x),\ \forall x \in [0,1].$

Proof : The monotonicity and the translation invariance of $GVaR_\lambda^\mu(\cdot)$ are a consequence of the definition of the lower quantile $r_{\cdot,\mu}^-(\lambda)$. The comonotonic additivity is due to proposition 1.3.1.
Let us now prove the representation formula (3.5.1). Schmeidler's representation theorem

(theorem 3.2.1) and remark 3.2.2 give the existence of a capacity ν on $(\Omega, \mathcal{F})$ such that $GVaR^{\mu}_{\lambda}(X) = \mathbb{E}_{\nu}(X),\ \forall X \in \chi$. For all $A \in \mathcal{F}$, we have

$$\nu(A) = GVaR^{\mu}_{\lambda}(\mathbb{I}_A) = r^{-}_{\mathbb{I}_A,\mu}(\lambda) = \mathbb{I}_{(1-\mu(A),1)}(\lambda).$$

Therefore, the capacity ν is of the form $\nu(A) = \psi(\mu(A)), \forall A \in \mathcal{F}$. The representation formula (3.5.1) is thus proved.
The representation result (3.5.1) being established, the property of consistency with respect to the $\leq_{mon,\mu}$ −relation follows from remark 3.3.8.

□

In general, the risk measure $GVaR^{\mu}_{\lambda}(\cdot)$ is not consistent with respect to the $\leq_{sl,\mu}$ −relation, the distortion function ψ in the representation formula (3.5.1) not being concave (cf. theorem 3.3.2 and remark 3.3.13).

Remark 3.5.1 In the previous definition/proposition the lower quantile $r^{-}_{\cdot,\mu}(\lambda)$ with respect to a given capacity μ at a given point λ is perceived as a "generalized" distortion risk measure (with respect to the capacity μ). An analogous result holds true for the upper quantile $r^{+}_{\cdot,\mu}(\lambda)$ thus providing another example of a "generalized" distortion risk measure. In the latter case, the distortion function ψ in the representation (3.5.1) has to be replaced by the function $x \longmapsto \mathbb{I}_{[1-\lambda,1]}(x)$.
We note that the risk measure $r^{+}_{\cdot,\mu}(\lambda)$ can be viewed as a generalization (to the case of a capacity) of the risk measure $Q^{+}_{\lambda}(\cdot)$ introduced in Dhaene et al. (2006).

Two particular cases are considered below - the case where the capacity μ is a distorted probability and the case where the capacity μ is an "upper envelope" of a given set of prior probability measures.

The case of a distorted probability

Let P be a given probability measure and ϕ be a given continuous distortion function. The first particular case which we consider is the case where the initial capacity μ is of the form $\mu = \phi \circ P$. The following result establishes a link, in this case, between the lower quantile function $r^{-}_{X,\mu}$ with respect to the capacity μ of a given measurable function X and the corresponding lower quantile function q^{-}_{X} with respect to the probability P.

Proposition 3.5.1 *Let P be a probability measure and ϕ be a given continuous distortion function. Let μ be a capacity of the form $\mu = \phi \circ P$. Let X be a given real-valued measurable function. Then, the following equality holds true for all $t \in (0,1)$:*

$$r^{-}_{X,\mu}(t) = q^{-}_{X}\left(1 - \check{\phi}(1-t)\right),$$

where $\check{\phi}$ denotes the upper generalized inverse of the non-decreasing function ϕ defined by $\check{\phi}(y) := \sup\{z : \phi(z) \leq y\},\ \forall y \in [0,1]$.

Proof : The proof of the previous proposition is placed in the appendix.

□

Remark 3.5.2 Under the assumptions of proposition 3.5.1, the following link between the *upper* quantile functions $r^+_{X,\mu}$ and q^+_X can be established :

$$r^+_{X,\mu}(t) = q^+_X(1 - \check{\phi}^-(1-t)),\ \forall t \in (0,1), \qquad (3.5.2)$$

where $\check{\phi}^-$ denotes the *lower* generalized inverse of the distortion function ϕ. The proof is based on arguments similar to those used in the proof of proposition 3.5.1 and is omitted. Let us note, however, that in the proof of the equality (3.5.2) we use the following equivalence, which is due to the assumption of continuity of the distortion function ϕ :

$$\phi(a) \geq t \text{ if and only if } a \geq \check{\phi}^-(t).$$

According to proposition 3.5.1, in the case where $\mu = \phi \circ P$ (and where the distortion function ϕ is continuous), the "generalized" Value at Risk with respect to μ at level $\lambda \in (0,1)$ is equal to the "classical" Value at Risk with respect to P at level $\tilde{\lambda}$ where $\tilde{\lambda} := 1 - \check{\phi}(1-\lambda)$.
One may wonder if the above relation between the risk measure $GVaR^{\mu}_{\lambda}$ and the risk measure $VaR_{\tilde{\lambda}}$ has an economic interpretation. Can the CEU-theory (upon which the motivation of the present chapter is based) explain the behaviour of an economic agent who, instead of assessing the risk of a given loss X by the Value at Risk of X at a level λ, assesses the risk of X by the Value at Risk of X at the (possibly different) level $\tilde{\lambda}$?
The measurable functions on $(\Omega, \mathcal{F})$ in the present chapter being interpreted as *losses*, we will consider an economic agent (an insurer, for instance) who is a CEU-*minimizer*. The agent's preferences are described by a "pain" function u and a capacity μ which, in the particular case that we consider, is of the form $\mu = \phi \circ P$.[3] The agent's "dissatisfaction" of a loss X is then assessed by the Choquet integral of $u(X)$ with respect to the capacity μ. When interpreting proposition 3.5.1 we will focus on three particular sub-cases : the case where there is "no distortion", the case of a concave (continuous) distortion ϕ, and the case of a convex (continuous) distortion ϕ. Let us remark that when an agent who is a CEU-*minimizer* is considered, the concavity (resp. the convexity) of the distortion function ϕ

3. In the case where the capacity μ is a distorted probability, the CEU-theory coincides with the so-called Rank-Dependent Expected Utility theory - see, for instance, Wang and Yan (2007) for a review.

is interpreted in terms of the agent's being a pessimist (resp. an optimist) [4].

1. The sub-case of a distortion function ϕ of the form $\phi(x) := x, \ \forall x \in [0,1]$
   In this sub-case we have $\tilde{\lambda} := 1 - \check{\phi}(1-\lambda) = \lambda$ where $\lambda \in (0,1)$ is a given level. This equality and proposition 3.5.1 lead to $GVaR^{\mu}_{\lambda}(X) = VaR_{\tilde{\lambda}}(X) = VaR_{\lambda}(X), \forall X \in \chi$. Hence, in the sub-case where the probability of events is perceived objectively (i.e. $\phi = id$), the risk measure $GVaR^{\mu}_{\lambda}$ at level $\lambda \in (0,1)$ is equal to the "usual" VaR_{λ} at the same level λ. We thus recover, by means of proposition 3.5.1, an observation which can be derived from the definitions of the two risk measures.
2. The sub-case of a concave (continuous) distortion function ϕ
   Let $\lambda \in (0,1)$ be a given level. The concavity of ϕ implies that $\tilde{\lambda} := 1 - \check{\phi}(1-\lambda) \geq \lambda$. Therefore, $VaR_{\tilde{\lambda}}(X) \geq VaR_{\lambda}(X), \forall X \in \chi$. By combining this inequality with proposition 3.5.1 we obtain that $GVaR^{\mu}_{\lambda}(X) = VaR_{\tilde{\lambda}}(X) \geq VaR_{\lambda}(X), \forall X \in \chi$. Thus, in the case where the agent is pessimistic (the distortion function ϕ being concave), the risk attributed to a loss X by means of the $GVaR^{\mu}_{\lambda}(X)$ is higher than (or equal to) the risk, equal to $VaR_{\lambda}(X)$, the agent would have attributed if he/ she had perceived events objectively without distorting them.
3. The sub-case of a convex (continuous) distortion function ϕ
   The convexity of ϕ implies that $\tilde{\lambda} := 1 - \check{\phi}(1-\lambda) \leq \lambda$. Therefore, in this sub-case, the inequality $GVaR^{\mu}_{\lambda}(X) \leq VaR_{\lambda}(X)$ holds for all $X \in \chi$.
   The risk $GVaR^{\mu}_{\lambda}(X)$ attributed by an optimistic agent (the distortion function ϕ in this sub-case being convex) to a given loss X is lower than (or equal to) the risk $VaR_{\lambda}(X)$ attributed to X by an agent who is objective.

An analogous reasoning applies to the risk measure $r^{+}_{\cdot,\mu}(\lambda)$; remark 3.5.2 is in this case used in place of proposition 3.5.1.

The case where μ is the upper envelope of a given set $\mathcal{P}$ of probability measures

We place oureselves in the context of model-uncertainty, expressed by a given non-empty set $\mathcal{P}$ of prior probability measures. The following result holds true.

Proposition 3.5.2 *Let $\mathcal{P}$ be a given non-empty set of probability measures on $(\Omega, \mathcal{F})$. Let us define a capacity μ on $(\Omega, \mathcal{F})$ by $\mu(A) := \sup_{P\in\mathcal{P}} P(A)$ and let X be a given real-valued measurable function on $(\Omega, \mathcal{F})$. Then, for all $t \in (0,1)$,*

$$r^{-}_{X,\mu}(t) = \sup_{P\in\mathcal{P}} q^{-}_{X,P}(t), \tag{3.5.3}$$

4. The situation considered more frequently in the literature (cf. Wang and Yan 2007, or Carlier and Dana 2003) is that of CEU-maximizers (the measurable functions on $(\Omega, \mathcal{F})$ being often interpreted as gains, instead of losses), in which case the interpretation of the concavity (resp. convexity) of the distortion function ϕ in terms of pessimism (resp. optimism) is reversed.

where $q^{-}_{X,P}$ denotes the lower quantile function of X with respect to the probability P.

Proof : The proof of proposition 3.5.2 is given in the appendix.

□

If the capacity μ of the form $\mu(\cdot) := \sup_{P\in\mathcal{P}} P(\cdot)$ is interpreted as expressing a pessimistic attitude towards model-uncertainty [5], the relation (3.5.3) of the previous proposition can be loosely interpreted as follows :
the risk, equal to $GVaR^{\mu}_{\lambda}(X)$, attributed to a given loss X by a pessimistic agent facing model-uncertainty, is equal to the supremum of the risks $VaR^{P}_{\lambda}(X)$ attributed to the loss X in each of the prior models $P \in \mathcal{P}$.

Remark 3.5.3 In the case where the capacity μ is the "lower envelope" of the set $\mathcal{P}$ of prior probability measures (i.e. $\mu(\cdot) := \inf_{P\in\mathcal{P}} P(\cdot)$) the following result about upper quantile functions can be shown :

$$r^{+}_{X,\mu}(t) = \inf_{P\in\mathcal{P}} q^{+}_{X,P}(t),\ \forall t \in (0,1).$$

An interpretation in terms of the agent's optimism could be given in this case.

Remark 3.5.4 In the case where $\mu(\cdot) := \sup_{P\in\mathcal{P}} P(\cdot)$ the "generalized" Value at Risk defined in definition/proposition 3.5.1 of the present chapter can be linked to a risk measure introduced in definition III.15 of Kervarec (2008). More precisely, the "generalized" Value at Risk $GVaR^{\mu}_{\lambda}(X)$ at level $\lambda \in (0,1)$ of a given measurable function $X \in \chi$ is equal to Kervarec's "Value at Risk" at level $(1-\lambda)$ of the measurable function $(-X)$. Indeed, thanks to the above proposition 3.5.2 and to lemma 2.1 in Dhaene et al. (2006), we obtain $GVaR^{\mu}_{\lambda}(X) = \sup_{P\in\mathcal{P}} -q^{+}_{-X,P}(1-\lambda)$. The term on the right-hand side of the previous equality is equal to Kervarec's "Value at Risk" at level $(1-\lambda)$ of $(-X)$ by proposition III.17 in Kervarec (2008) ; the desired link between the two risk measures is thus established.
We note as well that the minus sign preceding X in this relation is not surprising as the measurable functions on $(\Omega, \mathcal{F})$ in the present chapter are viewed as losses, whereas in the work of Kervarec (2008) they are perceived as gains.

5. Our interpretation of the capacity μ of the form $\mu(\cdot) := \sup_{P\in\mathcal{P}} P(\cdot)$ as expressing a pessimistic attitude towards model-uncertainty is motivated by the following observation : $\mathbb{E}_{\mu}(u(X)) \geq \mathbb{E}_{P}(u(X))$, $\forall X \in \chi$, $\forall P \in \mathcal{P}$, where $u : \mathbb{R} \to \mathbb{R}$ is a given Borel function. The inequality is due to proposition 5.2 (iii) in Denneberg (1994). A CEU-minimizer with a "pain" function u and a capacity $\mu(\cdot) := \sup_{P\in\mathcal{P}} P(\cdot)$ assesses his/her "dissatisfaction" with a loss $X \in \chi$ by the number $\mathbb{E}_{\mu}(u(X))$ which, according to the previous observation, is greater than (or equal to) the "dissatisfaction" $\mathbb{E}_{P}(u(X))$ associated to the loss X in any of the prior models $P \in \mathcal{P}$. Thus, in the context of model-uncertainty, a CEU-minimizer whose capacity μ is of the form $\mu(\cdot) = \sup_{P\in\mathcal{P}} P(\cdot)$ will be considered as being pessimistic.

3.5.2 A "generalized" Tail Value at Risk

The "classical" definition of the risk measure Tail Value at Risk is recalled hereafter for reader's convenience (cf., for instance, Dhaene et al. 2006). The "classical" Tail Value at Risk at level $\lambda \in (0,1)$ with respect to a given probability P of a given "potential loss" $X \in \chi$ (denoted by $TVaR_\lambda(X)$ or by $TVaR_\lambda^P(X)$) is defined by :

$$TVaR_\lambda(X) := \frac{1}{1-\lambda}\int_\lambda^1 q_X(t)dt$$

where the symbol q_X denotes a (version of the) quantile function of X with respect to the probability P. We note that the Tail Value at Risk of $X \in \chi$ at level $\lambda \in (0,1)$ (as defined above) is equal to the Average Value at Risk of $(-X)$ at level $(1-\lambda)$ appearing, for instance, in definition 4.43 of Föllmer and Schied (2004).
We consider hereafter a generalization of the previous definition to the case of a capacity which is not necessarily a probability measure. The definition and some properties of the "gerneralized" Tail Value at Risk are given in the following

Definition/Proposition 3.5.2 *Let μ be a capacity and let $\lambda \in (0,1)$.*
The functional $GTVaR_\lambda^\mu : \chi \to \mathbb{R}$ defined by $GTVaR_\lambda^\mu(X) := \frac{1}{1-\lambda}\int_\lambda^1 r_X^+(t)dt$, $\forall X \in \chi$ can be represented in the form :

$$GTVaR_\lambda^\mu(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi \tag{3.5.4}$$

where ψ is a concave distortion function given by $\psi(x) := \psi_\lambda(x) := \frac{1}{1-\lambda}\min\{1-\lambda; x\}$, $\forall x \in [0,1]$.
In particular, if μ is a concave capacity, $GTVaR_\lambda^\mu$ is a sub-additive functional on χ i.e. $GTVaR_\lambda^\mu(X+Y) \leq GTVaR_\lambda^\mu(X) + GTVaR_\lambda^\mu(Y), \forall X, Y \in \chi$.

Remark 3.5.5 The last statement in the previous definition/proposition 3.5.2 corresponds to exercise 6.7 in Denneberg (1994). The formulation given above is suitable for the needs of the present chapter.

Remark 3.5.6 The factor $\frac{1}{1-\lambda}$ in the definition of the functional $GTVaR_\lambda^\mu$ is necessary to obtain a normalized set function $\psi \circ \mu$ in the representation formula (3.5.4) in the sense that $\psi(\mu(\Omega)) = 1$.

Let us now prove the result ; the proof is based on lemma 3.3.1.
Proof : It is easy to check that the functional $GTVaR_\lambda^\mu$ satisfies properties (i), (ii) and (iii) of lemma 3.3.1 ; it follows, in particular, that there exists a non-decreasing function ψ defined on the set $S := \{\mu(A),\ A \in \mathcal{F}\}$ such that the representation (3.5.4) holds, namely

$GTVaR_\lambda^\mu(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi$. The expression of the function ψ on the set S can be computed from (3.5.4) as follows : for all $A \in \mathcal{F}$

$$\begin{aligned}\psi \circ \mu(A) = GTVaR_\lambda^\mu(\mathbb{I}_A) &= \frac{1}{1-\lambda}\int_\lambda^1 \mathbb{I}_{[1-\mu(A),1)}(t)dt \\ &= \frac{1}{1-\lambda}\Big(1 - \max\{\lambda; 1-\mu(A)\}\Big) = \frac{1}{1-\lambda}\min\{1-\lambda; \mu(A)\}.\end{aligned}$$

Then, ψ is extended to the whole interval $[0,1]$ by setting $\psi(x) := \frac{1}{1-\lambda}\min\{1-\lambda; x\},\ \forall x \in [0,1]$. The function ψ is obviously a concave distortion function.
In the case where μ is a concave capacity, the distorted capacity $\psi\circ\mu$ in the representation (3.5.4) is concave as the distortion function ψ is concave (see example 2.2.1 of chapter 2). The representation (3.5.4) and the property of sub-additivity of the Choquet integral with respect to a concave capacity allow us to conclude that the functional $GTVaR_\lambda^\mu$ is sub-additive in this case.

□

Thanks to the representation formula (3.5.4) of the previous definition/proposition and to theorem 3.3.3 we conclude that the functional $GTVaR_\lambda^\mu$ is a monetary risk measure on χ having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation.

Remark 3.5.7 We note that the monetary risk measure $GTVaR_\lambda^\mu$ can be used to characterize the $\leq_{sl,\mu}$ −stochastic dominance relation with respect to a capacity μ. More precisely, it follows from proposition 3.2.1 that :

$$X \leq_{sl,\mu} Y \text{ if and only if } GTVaR_\lambda^\mu(X) \leq GTVaR_\lambda^\mu(Y),\ \forall\lambda \in (0,1),$$

where X and Y are real-valued measurable functions such that $\int_0^1 |r_{X,\mu}(t)|dt < +\infty$ and $\int_0^1 |r_{Y,\mu}(t)|dt < +\infty$. The previous equivalence can be seen as a generalization to the case of a capacity of remark 4.44 in Föllmer and Schied (2004).

3.A Appendix

The counter-example of remark 3.4.1 :
Indeed, let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Let $\psi(x) := x^\alpha, \forall x \in [0,1]$ and $\phi(x) := x^\beta, \forall x \in [0,1]$ where $\alpha \in (0,1)$ and $\beta > \frac{1}{\alpha}$. Let us define a capacity μ by $\mu := \phi \circ P$ and a functional ρ by $\rho(X) := \mathbb{E}_{\psi\circ\mu}(X)$, $\forall X \in \chi$. The space $(\Omega, \mathcal{F}, \mu)$ satisfies the assumptions of theorem 3.4.3. Moreover, by applying theorem 3.3.3 (the distortion function ψ being concave), we obtain that the functional ρ is a monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the

$\leq_{sl,\mu}$ −relation. However, the functional ρ is not convex. The lack of convexity of ρ can be deduced from the fact that ρ can be represented as a Choquet integral with respect to the distorted probability $(\psi \circ \phi) \circ P$ where $\psi \circ \phi$ is a distortion function which is not concave (cf. proposition 4.69 and theorem 4.88 in Föllmer and Schied 2004).

The counter-example of remark 3.4.2 :

In the framework of the counter-example of subsection 3.3.4, let us define a functional $\rho : \chi \to \mathbb{R}$ by $\rho(X) = \mathbb{E}_{\psi\circ\mu}(X), \forall X \in \chi$ where the distortion function ψ and the capacity μ are the same as in the counter-example of subsection 3.3.4. We note that the space $(\Omega, \mathcal{F}, \mu)$ of the counter-example of subsection 3.3.4 satisfies the assumptions of theorem 3.4.3. Being a generalized distortion risk measure, the functional ρ satisfies the properties of comonotonic additivity and consistency with respect to the $\leq_{mon,\mu}$ −relation (cf. remark 3.3.8). Moreover, the capacity $\psi \circ \mu$ being concave, the functional ρ is convex. However, ρ is not consistent with respect to the $\leq_{sl,\mu}$ −relation as the distortion function ψ is not concave. The last statement can be easily deduced from theorem 3.3.2 and remark 3.3.13.

Proof of proposition 3.5.1 : Using the definition of the lower quantile function $r^-_{X,\mu}$ and the definition of the distribution function $G_{X,\mu}$, as well as the particular form of the capacity μ, we compute

$$\begin{aligned} r^-_{X,\mu}(t) &= \sup\{x \in \mathbb{R} : G_{X,\mu}(x) < t\} = \\ &= \sup\{x \in \mathbb{R} : \mu(X > x) > 1 - t\} = \\ &= \sup\{x \in \mathbb{R} : \phi(P(X > x)) > 1 - t\}. \end{aligned}$$

Now, the function ϕ being continuous by assumption, the following equivalence holds true

$$\phi(a) \leq t \text{ if and only if } a \leq \check{\phi}(t). \tag{3.A.1}$$

This observation implies that

$$\sup\{x \in \mathbb{R} : \phi(P(X > x)) > 1 - t\} = \sup\{x \in \mathbb{R} : P(X > x) > \check{\phi}(1 - t)\}.$$

Finally, it follows from the definition of the distribution function (with respect to P) F_X and the definition of the lower quantile function (with respect to P) q^-_X that

$$\begin{aligned} \sup\{x \in \mathbb{R} : P(X > x) > \check{\phi}(1 - t)\} &= \sup\{x \in \mathbb{R} : F_X(x) < 1 - \check{\phi}(1 - t)\} = \\ &= q^-_X(1 - \check{\phi}(1 - t)) \end{aligned}$$

which concludes the proof.

□

Proof of proposition 3.5.2 : Let $t \in (0,1)$. The definitions of the lower quantile function $r^-_{X,\mu}$ and of the distribution function $G_{X,\mu}$, as well as the particular form of the capacity μ lead to the following equalities :

$$\begin{aligned} r^-_{X,\mu}(t) &= \sup\{x \in \mathbb{R} : G_{X,\mu}(x) < t\} = \\ &= \sup\{x \in \mathbb{R} : 1 - \sup_{P \in \mathcal{P}} P(X > x) < t\} = \\ &= \sup\{x \in \mathbb{R} : \inf_{P \in \mathcal{P}} (1 - P(X > x)) < t\}. \end{aligned}$$

Therefore,

$$r^-_{X,\mu}(t) = \sup\{x \in \mathbb{R} : \inf_{P \in \mathcal{P}} F_{X,P}(x) < t\},$$

where $F_{X,P}$ denotes the distribution function of X with respect to the probability P. In order to establish the desired result, it suffices to prove that

$$\sup\{x \in \mathbb{R} : \inf_{P \in \mathcal{P}} F_{X,P}(x) < t\} = \sup_{P \in \mathcal{P}} q^-_{X,P}(t). \tag{3.A.2}$$

Let us first prove the inequality $\sup\{x \in \mathbb{R} : \inf_{P \in \mathcal{P}} F_{X,P}(x) < t\} \leq \sup_{P \in \mathcal{P}} q^-_{X,P}(t)$. Let $x \in \mathbb{R}$ be such that $\inf_{P \in \mathcal{P}} F_{X,P}(x) < t$. Then, there exists $P_x \in \mathcal{P}$ such that $F_{X,P_x}(x) < t$. This inequality and the definition of the lower quantile function q^-_{X,P_x} lead to $x \leq q^-_{X,P_x}(t)$. Thus, $x \leq \sup_{P \in \mathcal{P}} q^-_{X,P}(t)$.
Let us prove the converse inequality, namely $\sup\{x \in \mathbb{R} : \inf_{P \in \mathcal{P}} F_{X,P}(x) < t\} \geq \sup_{P \in \mathcal{P}} q^-_{X,P}(t)$. Let $P \in \mathcal{P}$ and let $x_P \in \mathbb{R}$ be such that $F_{X,P}(x_P) < t$. Then, x_P satisfies $\inf_{Q \in \mathcal{P}} F_{X,Q}(x_P) < t$. Therefore, $x_P \leq \sup\{y \in \mathbb{R} : \inf_{Q \in \mathcal{P}} F_{X,Q}(y) < t\}$. This inequality and the definition of the lower quantile function $q^-_{X,P}$ imply $q^-_{X,P}(t) \leq \sup\{y \in \mathbb{R} : \inf_{Q \in \mathcal{P}} F_{X,Q}(y) < t\}$. The probability $P \in \mathcal{P}$ being arbitrary, the proof is thus conluded.

$\square$

3.B Appendix

A closer look at the arguments of the proof of lemma 3.3.1 leads to the following observation :
A Choquet integral $\mathbb{E}_\nu(\cdot)$ with respect to a capacity ν satisfies the property (i) in lemma 3.3.1 if, and only if, the capacity ν and the initial capacity μ are linked in the following manner :

$$\mu(A) \leq \mu(B) \Rightarrow \nu(A) \leq \nu(B). \tag{3.B.1}$$

More precisely, the following proposition holds :

Proposition 3.B.1 *Let μ and ν be two capacities on $(\Omega, \mathcal{F})$. The following four statements are equivalent :*

(i) *The capacities μ and ν staisfy the property*

$$\mu(A) \leq \mu(B) \Rightarrow \nu(A) \leq \nu(B).$$

(ii) *There exists a distortion function ψ such that $\nu = \psi \circ \mu$.*

(iii) *There exists a distortion function ψ such that $\mathbb{E}_\nu(\cdot) = \mathbb{E}_{\psi \circ \mu}(\cdot)$.*

(iv) *The functional $\mathbb{E}_\nu(\cdot)$ satisfies the property :*

$$G_{X,\mu}(x) \geq G_{Y,\mu}(x), \forall x \in \mathbb{R} \Rightarrow \mathbb{E}_\nu(X) \leq \mathbb{E}_\nu(Y).$$

Proof : The proof of the implication $(i) \Rightarrow (ii)$ is contained in the proof of lemma 3.3.1. The implication $(ii) \Rightarrow (iii)$ is straightforward. The implication $(iii) \Rightarrow (iv)$ is shown in remark 3.3.7. In order to conclude, it remains to show that $(iv) \Rightarrow (i)$. Suppose that assertion (iv) holds. Let $A \in \mathcal{F}$ and $B \in \mathcal{F}$ be such that $\mu(A) \leq \mu(B)$. Then, $G_{\mathbb{I}_A,\mu}(x) \geq G_{\mathbb{I}_B,\mu}(x)$, for all $x \in \mathbb{R}$. This inequality, combined with (iv), gives $\mathbb{E}_\nu(\mathbb{I}_A) \leq \mathbb{E}_\nu(\mathbb{I}_B)$. Thus, we obtain $\nu(A) \leq \nu(B)$, which proves the desired implication.

□

3.C Appendix

In this appendix we provide more details on remark 3.3.4. More precisely, we recall theorem 2.2 statement (a) of Scarsini (1992), and we show in detail that remark 3.3.3 provides a counter-example to that statement. The following theorem recalls theorem 2.2 statement (a) (the case $n = 1$) of Scarsini (1992) by using the notation of the present thesis.

Theorem 3.C.1 (Scarsini) *Let ν_1, ν_2 be two capacities on $(\mathbb{R}, Bor(\mathbb{R}))$. The following assertions are equivalent :*

1. *$\mathbb{E}_{\bar{\nu}_1}(u) \geq \mathbb{E}_{\bar{\nu}_2}(u)$, for all $u : \mathbb{R} \to \mathbb{R}$ non-decreasing, provided the Choquet integrals exist in $\mathbb{R}$.*
2. *$\nu_1((-\infty, x]) \leq \nu_2((-\infty, x])$, for all $x \in \mathbb{R}$.*

Let the capacity μ, the measurable functions X and Y, and the non-decreasing function $u : \mathbb{R} \to \mathbb{R}$ be those of remark 3.3.3. We have shown in remark 3.3.3 that $G_{X,\mu}(x) \leq G_{Y,\mu}(x)$ and $\mathbb{E}_\mu(u(X)) < \mathbb{E}_\mu(u(Y))$.
Let us set $\nu_1 := \bar{\mu} \circ X^{-1}$, and $\nu_2 := \bar{\mu} \circ Y^{-1}$.
We note that ν_1 and ν_2 are capacities on $(\mathbb{R}, Bor(\mathbb{R}))$.
We note, moreover, that $\nu_1((-\infty, x]) = \bar{\mu}(X \leq x) = 1 - \mu(X > x) = G_{X,\mu}(x)$. Similarly,

$\nu_2((-\infty, x]) = G_{Y,\mu}(x)$.
On the other hand, we observe that, if $g : \mathbb{R} \to \mathbb{R}$ is a Borel function,

$$\mathbb{E}_{\bar{\nu}_1}(g) = \mathbb{E}_{\overline{\mu \circ X^{-1}}}(g) = \mathbb{E}_{\mu \circ X^{-1}}(g) = \mathbb{E}_{\mu}(g(X)), \tag{3.C.1}$$

where we have used the transformation rule for capacities (cf. Denneberg 1994, proposition 5.2) to obtain the last equality. By the same arguments, we have

$$\mathbb{E}_{\bar{\nu}_2}(g) = \mathbb{E}_{\mu}(g(Y)). \tag{3.C.2}$$

By applying equations (3.C.1) and (3.C.2) with $g := u$, we obtain $\mathbb{E}_{\bar{\nu}_1}(u) = \mathbb{E}_{\mu}(u(X))$ and $\mathbb{E}_{\bar{\nu}_2}(u) = \mathbb{E}_{\mu}(u(Y))$.
Thus, we have exhibited two capacities ν_1 and ν_2 on $(\mathbb{R}, Bor(\mathbb{R}))$, and a non-decreasing real-valued function u on $\mathbb{R}$ such that $\nu_1((-\infty, x]) \leq \nu_2((-\infty, x])$, for all $x \in \mathbb{R}$, and $\mathbb{E}_{\bar{\nu}_1}(u) < \mathbb{E}_{\bar{\nu}_2}(u)$, which provides the desired counter-example.

CHAPITRE 4

Robust representations of comonotonicly sub-additive and comonotonicly convex risk measures respecting a given generalized stochastic dominance relation

Abstract : In this chapter we establish "robust" representation results for the classes of monetary risk measures having the property of consistency with respect to the "generalized" increasing stochastic dominance relation and, either the properties of comonotonic sub-additivity and positive homogeneity, or, more generally, the property of comonotonic convexity. These classes of risk measures are represented in terms of maxima over a set of distortion functions of Choquet integrals with respect to a distorted capacity.
Keywords : Choquet integral, stochastic orderings with respect to a capacity, comonotonic sub-additivity, comonotonic convexity, distortion risk measure, quantile function with respect to a capacity, distorted capacity, Choquet expected utility, ambiguity, non-additive probability, behavioural finance, risk measurement, robust representation

4.1 Introduction

In the previous chapter we have been interested in monetary risk measures having the properties of comonotonic additivity and consistency with respect to a given "generalized" stochastic dominance relation. We will presently relax the property of comonotonic additivity to the property of comonotonic sub-additivity, and afterwords, to the property of comonotonic convexity. We recall the following definitions, where $(\Omega, \mathcal{F})$ is a given measurable space and χ is the space of bounded real-valued measurable functions on $(\Omega, \mathcal{F})$:

– A functional $\rho : \chi \to \mathbb{R}$ is called *comonotonicly sub-additive* if it satisfies the property :
  $\rho(X+Y) \leq \rho(X) + \rho(Y)$, for any pair (X, Y) of comonotonic measurable functions.
– A functional $\rho : \chi \to \mathbb{R}$ is called *comonotonicly convex* if it satisfies the property :
  $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, for any pair (X, Y) of comonotonic measurable functions, and any $\lambda \in (0, 1)$.

Monetary risk measures having the properties of positive homogeneity and comonotonic sub-additivity have been studied in the work of Laeven (2005), Song and Yan (2006), Heyde et al. (2007). The reader is also referred to Cont et al. (2010), and Cont et al. (2013) for a possible justification of the property of comonotonic sub-additivity from a different point of view. It is well-known that such risk measures could be seen as generalizations of the coherent risk measures of Artzner et al. (1999). Monetary risk measures having the property of comonotonic convexity have been studied by Song and Yan (2006). These risk measures could be seen as generalizing the notion of convex monetary risk measures in the sense of Föllmer and Schied (2004) (see also Frittelli and Rosazza Gianin 2002).
In Song and Yan (2009 a.) the authors have been interested in monetary risk measures defined on the space $L^\infty(\Omega, \mathcal{F}, P)$ having the properties of comonotonic sub-additivity, or comonotonic convexity, and consistency with respect to a given *"classical"* stochastic dominance relation with respect to the (initial) probability P. In particular, the following two results have been established by Song and Yan (2009 a.) (cf. theorems 3.1 and 3.5 respectively) :

Theorem 4.1.1 *Let P be a probability measure on $(\Omega, \mathcal{F})$. Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ be a monetary risk measure satisfying the properties of comonotonic sub-additivity, positive homogeneity and consistency with respect to the classical increasing stochastic dominance relation $\leq_{\mathrm{mon,P}}$. The functional ρ has the following representation :*

$$\rho(X) = \max_{\psi \in \mathcal{D}_P^\rho} \mathbb{E}_{\psi \circ P}(X), \text{ for all } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where $\mathcal{D}_P^\rho := \{\psi$ distortion function such that $\mathbb{E}_{\psi \circ P}(Y) \leq \rho(Y),$ for all $Y \in L^\infty(\Omega, \mathcal{F}, P)\}$.

Theorem 4.1.2 *Let P be a probability measure on $(\Omega, \mathcal{F})$. Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ be a monetary risk measure satisfying the properties of comonotonic convexity and consistency with respect to the classical increasing stochastic dominance relation $\leq_{\mathrm{mon,P}}$. The functional ρ has the following representation :*

$$\rho(X) = \max_{\psi \in \mathcal{D}} \left(\mathbb{E}_{\psi \circ P}(X) - \alpha(\psi)\right), \text{ for all } X \in L^\infty(\Omega, \mathcal{F}, P),$$

where $\mathcal{D}$ denotes the set of distortion functions, and $\alpha(\cdot)$ is a penalty function defined by $\alpha(\psi) := \alpha_P^\rho(\psi) := \sup_{\{Y \in L^\infty(P) : \rho(Y) \leq 0\}} \mathbb{E}_{\psi \circ P}(Y)$, for all $\psi \in \mathcal{D}$.

The purpose of the present chapter is to establish analogues of the two theorems recalled above in the setting where the underlying measurable space $(\Omega, \mathcal{F})$ is endowed with a given capacity μ which is not necessarily a probability measure, and where the requirement of consistency with respect to the *"classical"* increasing stochastic dominance is replaced by the requirement of consistency with respect to the "generalized" increasing stochastic dominance with respect to the capacity μ.
The remainder of the present chapter is organized as follows : In section 4.2 we state and prove the analogues of the above theorems in the case of a capacity. In section 4.3 we briefly mention some perspectives for future research. In the appendix we give two lemmas which are used in the proofs of the theorems from section 4.2.

4.2 "Robust" representations of comonotonicly sub-additive and comonotonicly convex risk measures respecting a given "generalized" stochastic dominance relation

In the following theorem we establish a representation result for the class of monetary risk measures on χ having the properties of comonotonic sub-additivity, positive homogeneity and consistency with respect to the $\leq_{mon,\mu}$ −stochastic dominance relation, where μ is a given capacity assumed to be continuous from below and from above.

Theorem 4.2.1 *Let μ be a capacity on $(\Omega, \mathcal{F})$ which is continuous from below and from above. Let $\rho : \chi \to \mathbb{R}$ be a monetary risk measure satisfying the properties of comonotonic sub-additivity, positive homogeneity and consistency with respect to the $\leq_{\mathrm{mon},\mu}$ −stochastic dominance relation. The functional ρ has the following representation :*

$$\rho(X) = \max_{\psi \in \mathcal{D}^\rho} \mathbb{E}_{\psi \circ \mu}(X), \text{ for all } X \in \chi,$$

where $\mathcal{D}^\rho := \{\psi \text{ distortion function such that } \mathbb{E}_{\psi\circ\mu}(Y) \leq \rho(Y), \text{ for all } Y \in \chi\}$.

The above theorem is the analogue of theorem 3.1 of Song and Yan (2009 a.). Its proof is based on an adaptation of the arguments of Song and Yan (2009 a.) to the present setting, as well as on some specific arguments relating to capacities. The following lemma will be used in the proof of the theorem.

Lemma 4.2.1 *Let μ a capacity on $(\Omega, \mathcal{F})$ which is continuous from below and from above. Let X_1, X_2, Y_1 and Y_2 be four real-valued measurable functions on $(\Omega, \mathcal{F})$ such that X_1 and X_2 are comonotonic, Y_1 and Y_2 are comonotonic, and $X_1 \leq_{\mathrm{mon},\mu} Y_1$, $X_2 \leq_{\mathrm{mon},\mu} Y_2$. Then, $X_1 + X_2 \leq_{\mathrm{mon},\mu} Y_1 + Y_2$.*

The lemma is well-known in the classical case where μ is a probability measure (cf. lemma 2.2 in Song and Yan 2009 a. and the references therein).

Proof of lemma 4.2.1 : Thanks to the characterization of the $\leq_{mon,\mu}$ −relation (proposition 3.3.1), it suffices to prove that $r^+_{X_1+X_2}(t) \leq r^+_{Y_1+Y_2}(t)$, for all $t \in (0,1)$. The measurable functions X_1 and X_2 being comonotonic, we have, in virtue of proposition 1.3.1, that

$$r^+_{X_1+X_2}(t) = r^+_{X_1}(t) + r^+_{X_2}(t), \text{ for all } t \in (0,1). \tag{4.2.1}$$

Similarly,

$$r^+_{Y_1+Y_2}(t) = r^+_{Y_1}(t) + r^+_{Y_2}(t), \text{ for all } t \in (0,1). \tag{4.2.2}$$

As $X_1 \leq_{mon,\mu} Y_1$ and $X_2 \leq_{mon,\mu} Y_2$, we have $r^+_{X_1}(t) \leq r^+_{Y_1}(t)$, for all $t \in (0,1)$, and $r^+_{X_2}(t) \leq r^+_{Y_2}(t)$, for all $t \in (0,1)$. These inequalities combined with equations (4.2.1) and (4.2.2) give the desired conclusion.

□

Remark 4.2.1 Before giving the proof of theorem 4.2.1, we note that any functional $\rho : \chi \to \mathbb{R}$ satisfying the properties of theorem 4.2.1 is normalized in the following sense : $\rho(\mathbb{I}) = 1$.

Proof of theorem 4.2.1 : By definition of the set $\mathcal{D}^\rho$, we have $\rho(X) \geq \sup_{\psi \in \mathcal{D}^\rho} \mathbb{E}_{\psi\circ\mu}(X)$, for all $X \in \chi$. In order to prove the desired result, it suffices to prove that, for any $X \in \chi$, there exists a distortion function $\psi_X \in \mathcal{D}^\rho$ such that $\rho(X) = \mathbb{E}_{\psi_X\circ\mu}(X)$. To do that, we will use the separation theorem of Hahn-Banach.

Let $X \in \chi$. The functional ρ and the Choquet integral being translation invariant, we can consider, without loss of generality, the case where $\rho(X) = 1$. Similarly to Song and Yan (2009 a.), we denote by $[X]$ the set of measurable functions of the form $u(X)$ where u is a non-decreasing continuous function on $\mathbb{R}$. We denote by $ba(\Omega, \mathcal{F})$ the space of finitely additive set functions on $(\Omega, \mathcal{F})$ whose total variation is finite. We recall that the space $ba(\Omega, \mathcal{F})$ can be identified with the dual space of $(\chi, \|\cdot\|)$, where $\|\cdot\|$ denotes the supremum norm on χ. Similarly to Song and Yan (2009 a.), we set $\mathcal{B} := \{Y \in \chi : \exists Z \in [X] \text{ with } \rho(Z) < 1 \text{ and } Y \leq Z\}$. We check that $X \notin \mathcal{B}$. By using the property of comonotonic sub-additivity of ρ, we check that $\mathcal{B}$ is a convex set. Moreover, we can check that the set $\mathcal{B}_1$, defined by $\mathcal{B}_1 := \{Y \in \chi : \|Y\| < 1\}$, is included in $\mathcal{B}$. Indeed, let $Y \in \chi$ be such that $\|Y\| < 1$ (i.e. $Y \in \mathcal{B}_1$). We have $\|Y\| \in [X]$, $Y \leq \|Y\|$, and $\rho(\|Y\|) = \|Y\|\rho(\mathbb{I}) = \|Y\| < 1$, where we have used remark 4.2.1. The inclusion $\mathcal{B}_1 \subset \mathcal{B}$ is thus proved. This inclusion implies that the interior of $\mathcal{B}$ is not empty.

By the separation theorem of Hahn-Banach, there exists a non-trivial $\lambda \in ba(\Omega, \mathcal{F})$ such that $\sup_{Y\in\mathcal{B}} \lambda(Y) \leq \lambda(X)$, where we denote by the same symbol the set function λ, as well as the corresponding functional. The functional λ satisfies $\lambda(X) > 0$. Therefore, we can choose $\lambda \in ba(\Omega, \mathcal{F})$ such that $\lambda(X) = 1$. We can show that λ has the following properties :

1. $\lambda(Y) \geq 0$, for all $Y \in \chi_+$.
2. $\lambda(\mathbb{I}) = 1$.
3. $\lambda(Y) \leq \rho(Y)$, for all $Y \in [X]$.

The above properties can be proved by using the same arguments as those of Song and Yan (2009 a.) ; the proof is given in lemma 4.A.1 of the appendix for reader's convenience. We define the functional ρ_* on χ by :

$$\rho_*(Y) := \sup\{\lambda(Z) : Z \in [X], Z \leq_{mon,\mu} Y\}, \text{ for all } Y \in \chi.$$

We denote, for the easing of the presentation, $\mathcal{E}^{mon}_{X,Y} := \{Z \in \chi : Z \in [X], Z \leq_{mon,\mu} Y\}$. It can be checked that ρ_* is real-valued.
The functional ρ_* has the following properties :

1. The functional ρ_* is positively homogeneous (to prove this property, if suffices to remark that, for all $a > 0$, $Z \leq_{mon,\mu} aY$ if and only if $\frac{1}{a}Z \leq_{mon,\mu} Y$).
2. The functional ρ_* is translation invariant (to prove this property, if suffices to remark that, for all $b \in \mathbb{R}$, $Z \leq_{mon,\mu} Y + b$ if and only if $Z - b \leq_{mon,\mu} Y$).
3. The functional ρ_* is consistent with respect to the $\leq_{mon,\mu}$ −relation (this property is due to the transitivity of the $\leq_{mon,\mu}$ −relation on χ).
4. $\rho_*(Y) \leq \rho(Y)$, for all $Y \in \chi$. This property is due to the definition of ρ_*, to property 3 of λ, and to the property of consistency with respect to the $\leq_{mon,\mu}$ −relation of ρ.
5. $\rho_*(Y) \geq \lambda(Y)$, for all $Y \in [X]$. This property is due to the fact that $Y \in \mathcal{E}^{mon}_{X,Y}$, for all $Y \in [X]$, and to the definition of ρ_*.
6. The functional ρ_* has the property of comonotonic additivity.
   To prove this property, let Y_1 and Y_2 be two comonotonic measurable functions in χ. Thanks to proposition 1.2.1 , there exist two non-decreasing continuous functions u and v on $\mathbb{R}$ such that $u + v = id$ and $Y_1 = u(Y_1 + Y_2)$, $Y_2 = v(Y_1 + Y_2)$. Let us show first the inequality $\rho_*(Y_1 + Y_2) \leq \rho_*(Y_1) + \rho_*(Y_2)$. Let $Z \in \mathcal{E}^{mon}_{X,Y_1+Y_2}$. As $u + v = id$, we have $Z = u(Z) + v(Z)$. Thus,

$$\lambda(Z) = \lambda(u(Z)) + \lambda(v(Z)). \tag{4.2.3}$$

   We note that

$$u(Z) \in \mathcal{E}^{mon}_{X,Y_1} \text{ and } v(Z) \in \mathcal{E}^{mon}_{X,Y_2}. \tag{4.2.4}$$

   Indeed, since $Z \in \mathcal{E}^{mon}_{X,Y_1+Y_2}$, we have $Z \in [X]$, which means that there exists a non-decreasing continuous function $w : \mathbb{R} \to \mathbb{R}$ such that $Z = w(X)$. By composition, the functions $u \circ w$ and $v \circ w$ are non-decreasing continuous, which gives $u(Z) \in [X]$ and $v(Z) \in [X]$. On the other hand, as $Z \leq_{mon,\mu} Y_1 + Y_2$ and u is non-decreasing, we have $u(Z) \leq_{mon,\mu} u(Y_1+Y_2)$. Analogously, $v(Z) \leq_{mon,\mu} v(Y_1+Y_2)$. Equation (4.2.4) is thus proved.

By using equations (4.2.4) and (4.2.3), and the definition of ρ_*, we obtain $\lambda(Z) \leq \rho_*(Y_1) + \rho_*(Y_2)$. The measurable function $Z \in \mathcal{E}^{mon}_{X,Y_1+Y_2}$ being arbitrary, we obtain $\rho_*(Y_1+Y_2) \leq \rho_*(Y_1) + \rho_*(Y_2)$.

Let us now prove the converse inequality, namely

$$\rho_*(Y_1+Y_2) \geq \rho_*(Y_1) + \rho_*(Y_2). \tag{4.2.5}$$

Let $Z_1 \in \mathcal{E}^{mon}_{X,Y_1}$ and $Z_2 \in \mathcal{E}^{mon}_{X,Y_2}$. We have $Z_1 + Z_2 \in [X]$ (since $Z_1 = w_1(X)$ and $Z_2 = w_2(X)$, where w_1 and w_2 are non-decreasing continuous functions on $\mathbb{R}$). Moreover, thanks to lemma 4.2.1, $Z_1+Z_2 \leq_{mon,\mu} Y_1+Y_2$. Hence, $Z_1+Z_2 \in \mathcal{E}^{mon}_{X,Y_1+Y_2}$. By using that observation and the definition of ρ_*, we get $\rho_*(Y_1+Y_2) \geq \lambda(Z_1+Z_2) = \lambda(Z_1)+\lambda(Z_2)$. The measurable functions $Z_1 \in \mathcal{E}^{mon}_{X,Y_1}$ and $Z_2 \in \mathcal{E}^{mon}_{X,Y_2}$ being arbitrary, we obtain the desired inequality (4.2.5). The comonotonic additivity of ρ_* is thus proved.

In virtue of the representation result of theorem 3.3.1 applied to the functional $\rho_* : \chi \to \mathbb{R}$ (which is translation invariant, comonotonicly additive and consistent with respect to the $\leq_{mon,\mu}$ −relation), there exists a distortion function $\psi_X : [0,1] \to [0,1]$ such that $\rho_*(Y) = \mathbb{E}_{\psi_X \circ \mu}(Y)$, for all $Y \in \chi$.

Thanks to property 4 of ρ_*, we have $\mathbb{E}_{\psi_X \circ \mu}(Y) \leq \rho(Y)$, for all $Y \in \chi$, which implies $\psi_X \in \mathcal{D}^\rho$. On the other hand, thanks to properties 4 and 5 of ρ_*, we have $1 = \lambda(X) \leq \rho_*(X) \leq \rho(X) = 1$, which leads to $\rho_*(X) = \rho(X) = 1$. We conclude that the distortion function ψ_X is as desired.

□

The following theorem is an analogue of theorem 3.5 in Song and Yan (2009 a.) (cf. also theorem 3.2 in Song and Yan 2009 b.).

Theorem 4.2.2 *Let μ be a capacity on $(\Omega, \mathcal{F})$ which is continuous from below and from above. Let $\rho : \chi \to \mathbb{R}$ be a monetary risk measure satisfying the properties of comonotonic convexity and consistency with respect to the $\leq_{\mathrm{mon},\mu}$ −stochastic dominance relation. The functional ρ has the following representation :*

$$\rho(X) = \max_{\psi \in \mathcal{D}} \left(\mathbb{E}_{\psi \circ \mu}(X) - \alpha(\psi) \right), \text{ for all } X \in \chi,$$

where $\mathcal{D}$ denotes the set of distortion functions, and $\alpha(\cdot)$ is a penalty function defined by $\alpha(\psi) := \sup_{\{Y \in \chi : \rho(Y) \leq 0\}} \mathbb{E}_{\psi \circ \mu}(Y)$, for all $\psi \in \mathcal{D}$.

In the proof of the above theorem we adapt the arguments of Song and Yan 2009 b.(theorem 3.2) to the present setting ; the proof is given for reader's convenience.

Proof of theorem 4.2.2 : We can assume, without loss of generality, that $\rho(0) = 0$. Before proceeding further, we note also that the penalty function $\alpha(\cdot)$ has its values in $\mathbb{R} \cup \{+\infty\}$.

Let us first show that, for all $X \in \chi$, $\rho(X) \geq \sup_{\psi \in \mathcal{D}} \left(\mathbb{E}_{\psi\circ\mu}(X) - \alpha(\psi)\right)$. Let $X \in \chi$. We set $X_1 := X - \rho(X)$ and we note that $\rho(X_1) = 0$, which is due to the translation invariance of ρ. Thus, for all $\psi \in \mathcal{D}$,

$$\alpha(\psi) := \sup_{\{Y \in \chi : \rho(Y) \leq 0\}} \mathbb{E}_{\psi\circ\mu}(Y) \geq \mathbb{E}_{\psi\circ\mu}(X_1) = \mathbb{E}_{\psi\circ\mu}(X) - \rho(X),$$

where we have used the property of translation invariance of the Choquet integral $\mathbb{E}_{\psi\circ\mu}(\cdot)$ to obtain the last equality. It follows that $\rho(X) \geq \sup_{\psi \in \mathcal{D}} \left(\mathbb{E}_{\psi\circ\mu}(X) - \alpha(\psi)\right)$.

It remains to show that for any $X \in \chi$, there exists $\psi_X \in \mathcal{D}$ such that $\rho(X) \leq \mathbb{E}_{\psi_X\circ\mu}(X) - \alpha(\psi_X)$. By translation invariance of the left-hand side and the right-hand side, it suffices to consider $X \in \chi$ such that $\rho(X) = 0$. We define the set $[X]$ as in the proof of the previous theorem 4.2.1. The set $\mathcal{B}$ is defined by $\mathcal{B} := \{Y \in \chi : \exists Z \in [X] \text{ with } \rho(Z) < 0 \text{ and } Y \leq Z\}$. We note that $X \notin \mathcal{B}$. By using the property of comonotonic convexity of ρ, we check that $\mathcal{B}$ is a convex set. Moreover, we can check that the set $\mathcal{B}_1$, defined by $\mathcal{B}_1 := \{Y \in \chi : \|Y + 1\| < 1\}$, is included in $\mathcal{B}$. Indeed, let $Y \in \chi$ be such that $\|Y + 1\| < 1$ (i.e. $Y \in \mathcal{B}_1$). We have $\|Y + 1\| - 1 \in [X]$, $Y = Y + 1 - 1 \leq \|Y + 1\| - 1$, and $\rho(\|Y + 1\| - 1) = \rho(0) + \|Y + 1\| - 1 = \|Y + 1\| - 1 < 0$. Hence, $Y \in \mathcal{B}$. The inclusion $\mathcal{B}_1 \subset \mathcal{B}$ implies that the interior of $\mathcal{B}$ is not empty.

By the separation theorem of Hahn-Banach, there exists a non-trivial $\lambda \in ba(\Omega, \mathcal{F})$ such that $b := \sup_{Y \in \mathcal{B}} \lambda(Y) \leq \lambda(X)$.

By arguments similar to those of Song and Yan (2009 b.), we show that λ has the following properties :

1. $\lambda(Y) \geq 0$, for all $Y \in \chi_+$.
   In order to prove this property, let $Y \in \chi_+$ and $c > 0$. We have $-1 - cY \in \mathcal{B}$. Therefore, $\lambda(-1 - cY) \leq \lambda(X)$. The functional $\lambda(\cdot)$ being linear, and the number $c > 0$ being arbitrary, we obtain $\lambda(Y) \geq 0$.

2. $\lambda(\mathbb{I}) > 0$. This property is due to lemma 4.A.2.

Thanks to property 2 we can choose λ such that $\lambda(\mathbb{I}) = 1$. We define the functional ρ_* on χ by :

$$\rho_*(Y) := \sup\{\lambda(Z) : Z \in [X], Z \leq_{mon,\mu} Y\}, \text{ for all } Y \in \chi.$$

It can be checked that ρ_* is real-valued.

The functional ρ_* has the following properties (which can be proved by arguments similar to those used in the proof of the previous theorem 4.2.1) :

1. The functional ρ_* is translation invariant.

2. The functional ρ_* is consistent with respect to the $\leq_{mon,\mu}$ −relation.

3. The functional ρ_* is comonotonicly additive.

We note moreover that

4. $\rho_*(X) \geq \lambda(X)$. This property is due to the definition of ρ_* and to the reflexivity of the $\leq_{mon,\mu}$ −relation.

In virtue of the representation result of theorem 3.3.1 applied to the functional $\rho_* : \chi \to \mathbb{R}$, there exists a distortion function $\psi_X : [0,1] \to [0,1]$ such that $\rho_*(Y) = \mathbb{E}_{\psi_X \circ \mu}(Y)$, for all $Y \in \chi$.
Let $Y \in \chi$ be such that $\rho(Y) \leq 0$. For all $\varepsilon > 0$,

$$\mathbb{E}_{\psi_X \circ \mu}(Y) - \varepsilon = \mathbb{E}_{\psi_X \circ \mu}(Y - \varepsilon) = \rho_*(Y - \varepsilon) = \sup\{\lambda(Z) : Z \in [X], Z \leq_{mon,\mu} Y - \varepsilon\}. \quad (4.2.6)$$

Now, for any $Z \in \chi$ such that $Z \in [X]$ and $Z \leq_{mon,\mu} Y - \varepsilon$, we have $Z \in \mathcal{B}$. This can be shown by observing that $\rho(Z) \leq \rho(Y - \varepsilon) = \rho(Y) - \varepsilon < 0$, where the first inequality is obtained thanks to the property of consistency with respect to the $\leq_{mon,\mu}$ −relation of ρ. Thus,

$$\sup\{\lambda(Z) : Z \in [X], Z \leq_{mon,\mu} Y - \varepsilon\} \leq \sup\{\lambda(Z) : Z \in \mathcal{B}\} = b. \quad (4.2.7)$$

By combining equations (4.2.6) and (4.2.7), we obtain $\mathbb{E}_{\psi_X \circ \mu}(Y) - \varepsilon \leq b$. The number $\varepsilon > 0$ being arbitrary, we get $\mathbb{E}_{\psi_X \circ \mu}(Y) \leq b$. The measurable function Y being arbitrarily chosen in the set $\{Y \in \chi : \rho(Y) \leq 0\}$, we obtain $\alpha(\psi_X) = \sup_{\{Y \in \chi : \rho(Y) \leq 0\}} \mathbb{E}_{\psi_X \circ \mu}(Y) \leq b$. Thus,

$$\mathbb{E}_{\psi_X \circ \mu}(X) - \alpha(\psi_X) \geq \mathbb{E}_{\psi_X \circ \mu}(X) - b = \rho_*(X) - b \geq \lambda(X) - b \geq 0 = \rho(X),$$

where we have used property 4 of ρ_*. The distortion function ψ_X is as desired.

□

4.3 Future complements

In the previous section 4.2 we have been interested in monetary risk measures having the properties of comonotonic sub-additivity or comonotonic convexity, and consistency with respect to the "generalized" *increasing* stochastic dominance relation. In order to complete our study, it remains to investigate the classes of monetary risk measures having the properties of comonotonic sub-additivity or comonotonic convexity, and consistency with respect to the "generalized" *increasing convex* stochastic dominance relation, which we will leave for our future research.

4.A Appendix

In this appendix we give two lemmas which are used in the proof of theorem 4.2.1 and of theorem 4.2.2, respectively. The proof of the following lemma is contained in the proof of theorem 3.1 in Song and Yan (2009 a.) and is recalled here for reader's convenience.

Lemma 4.A.1 (Song and Yan (2009 a.)) *Let $\rho : \chi \to \mathbb{R}$ be as in theorem 4.2.1 (in particular, $\rho(c) = c, \forall c \in \mathbb{R}$). Let $X \in \chi$. Let $[X]$ and $\mathcal{B}$ be the sets from theorem 4.2.1, i.e. $[X] := \{u(X) : u$ non-decreasing continuous$\}$ and $\mathcal{B} := \{Y \in \chi : \exists Z \in [X]$ with $\rho(Z) < 1$ and $Y \leq Z\}$. Let $\lambda \in ba(\Omega, \mathcal{F})$ be as in theorem 4.2.1, that is $\lambda(X) = 1$ and $\sup_{Y \in \mathcal{B}} \lambda(Y) \leq \lambda(X)$. Then, the functional λ has the following properties :*

1. *$\lambda(Y) \geq 0$, for all $Y \in \chi_+$.*
2. *$\lambda(\mathbb{I}) = 1$.*
3. *$\lambda(Y) \leq \rho(Y)$, for all $Y \in [X]$.*

Proof : In order to prove property 1, let $Y \in \chi_+$ and let $c > 0$. We have $-cY \leq 0$, and we see (by taking $Z \equiv 0$) that $-cY \in \mathcal{B}$. Hence, $\lambda(-cY) \leq \lambda(X) = 1$, which gives $\lambda(Y) \geq -\frac{1}{c}$. The number $c > 0$ being arbitrary, we obtain $\lambda(Y) \geq 0$.

Let us prove property 2. We show first that $\lambda(\mathbb{I}) \leq 1$. Let $c \in (0,1)$. As $c \in \mathcal{B}$, we have $\lambda(c) \leq \lambda(X) = 1$. Hence, $c\lambda(\mathbb{I}) \leq 1$, which gives $\lambda(\mathbb{I}) \leq \frac{1}{c}$. The number $c \in (0,1)$ being arbitrary, we get $\lambda(\mathbb{I}) \leq 1$. Let us show the converse inequality, namely $\lambda(\mathbb{I}) \geq 1$. Let $c > 1$. As $2 - c \in \mathcal{B}$, we have $\lambda(2-c) \leq \lambda(X) = 1$. Hence, $2 - c\lambda(\mathbb{I}) \leq 1$, which gives $\lambda(\mathbb{I}) \geq \frac{1}{c}$. The number $c > 1$ being arbitrary, we obtain the desired inequality.

In order to prove property 3, let $Y \in [X]$. We set $Y_1 := Y - \rho(Y) + 1$. Let $c > 1$. We check that $\frac{1}{c}Y_1 \in \mathcal{B}$. Thus, $\lambda(Y_1) \leq c$. The number $c > 1$ being arbitary, we get $\lambda(Y_1) \leq 1$. By using the linearity of λ and the definition of Y_1, we obtain $\lambda(Y) \leq \rho(Y)$.

□

The proof of the following lemma is contained in the proof of theorem 3.2 in Song and Yan (2009 b.) and is recalled here for reader's convenience.

Lemma 4.A.2 (Song and Yan (2009 b.)) *Let $\lambda : \chi \to \mathbb{R}$ be a non-trivial continuous linear functional on $(\chi, \|\cdot\|)$ satisfying the following property :*

– *(positivity) $\lambda(Y) \geq 0, \forall Y \in \chi_+$.*

Then, $\lambda(\mathbb{I}) > 0$.

Proof : The functional λ being non-trivial, there exists $Y \in \chi$ such that $\|Y\| \leq 1$ and $\lambda(Y) > 0$. Let us show that

(i) $\lambda(Y_+) > 0$ and

(ii) $\lambda(1 - Y_+) \geq 0$.

As $\lambda(Y) > 0$ and as λ is linear, we have $\lambda(Y) = \lambda(Y_+) - \lambda(Y_-) > 0$. On the other hand, $\lambda(Y_-) \geq 0$ thanks to the property of positivity of λ. Thus, we obtain $\lambda(Y_+) > \lambda(Y_-) \geq 0$, which proves (i).

Let us now prove property (ii). We have $Y_+ \leq |Y| \leq \|Y\| \leq 1$. Hence, $1 - Y_+ \geq 0$. Thanks to the positivity of λ we obtain $\lambda(1 - Y_+) \geq 0$, which proves (ii).

Combining properties (i) and (ii) gives $\lambda(\mathbb{I}) \geq \lambda(Y_+) > 0$.

$\square$

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